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Eric Stephen Barnes 1924–2000

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Formative Years 1924–39

Eric Barnes was born on 16 January 1924 in Cardiff, Wales. He was the only child of William H. and Dorothy Barnes. His father did not proceed beyond primary school and spent his life as a manual worker. His mother attended secondary school and took various clerical jobs. The family suffered during the depression of the 1920s in South Wales and this prompted the father to migrate to Sydney in 1926, and Eric and his mother followed in 1927. His mother could not tolerate the new life and went back with Eric to Cardiff, but they eventually returned to Sydney in 1929.

Eric proved to be a precocious child, learning to read when very young and able to tell the time at age 3. He attended Punchbowl Primary School and accelerated through the grades to the final grade at age 9. His results in the Primary Final examination qualified him for admission to a full high school but he was considered too young to proceed. He sat the Primary Final again the following year and finished high enough to qualify for a State Government Bursary; but alas there was a condition that one had to be 11 years old by the following 1 January, and the award was refused since Eric was 15 days too young. Nevertheless, with the support of his parents, he entered Canterbury Boys' High School in 1935. At the end of his first year he took the Primary Final for the third time and was duly awarded a Government Bursary.

Eric excelled academically at Canterbury. For the Intermediate Certificate in 1937 he gained the maximum of eight As (and was awarded a pair of gold cufflinks). At the Leaving Examination in 1939 he gained first-class honours in the two Mathematics subjects and French, second-class honours in German, and As in English and Latin. He was awarded the Barker Scholarship for Mathematics, shared the Garton Prize for French, and won a University Exhibition and a State Government Bursary.

University of Sydney 1940–43

Eric was first attracted to a career as an actuary. The arduous seven-year course was conducted from Britain. At the end of 1939 a torpedo sank the ship carrying the examination papers and the course was abandoned for 'the duration'. Eric opted for an Honours BA course in both Mathematics and French at the University of Sydney. He won prizes in both subjects every year, graduating at the beginning of 1943 with First-Class Honours in Mathematics and French. Although he was recommended for University Medals in both subjects, the recommendation was rejected because he had taken three and not four years for the honours courses.

War Service 1943–45

During 1942, Eric had been approached by A.D. Trendall, Professor of Greek at the University of Sydney, as a candidate for a special unit of the Australian Intelligence Corps located at Victoria Barracks in Melbourne. Eric duly joined the Citizen Military Forces in 1943 and served for three years in the Intelligence Section, with the

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E. S. Barnes May 1955.

particular task of decoding Japanese diplomatic messages.

It was only in the 1990s that information concerning the secret work of the Australian Intelligence Unit was made public. In an article 'Our War of Words' in the *Sydney Morning Herald* (19 September 1992), the author David Jenkins profiled the Australian code-breakers 'who helped change the course of history'. The code-breakers were mainly classicists and mathematicians. To quote from the article: 'There were, however, notable exceptions to the rule that said that classicists were better than mathematicians. One or two of the younger mathematicians, Barnes in particular, proved to be highly skilled code-breakers'. In conversation in later years Eric recalled that he gained his commission as a Lieutenant because of his success in cracking a Japanese code that had baffled the British experts at Bletchley Park.

An interesting account has been given by some of Eric's colleagues in the Special Intelligence Section of the solving of the so-called Kormoran cryptogram. Although there is some controversy about the matter, HMAS *Sydney* was apparently sunk, with all crew lost, by the German raider *Kormoran* off the coast of Western Australia on 19 November 1941. Captain Detmers and the crew of the damaged *Kormoran* abandoned their ship and were captured and interned. On 11 January 1945 Detmers, with others, escaped from a POW camp and when recaptured had in his possession a book with coded messages. These were sent to the Section and other Intelligence groups for decryption. One of Eric's colleagues writes: 'It was not difficult to break this cipher, once it had been recognised. I cannot remember the details of this breaking, but I am sure the crucial steps were taken by Barnes, who used to see in two minutes what would take me two hours.' The same colleague described Eric as having a 'laser-sharp mind'.

Although Eric found his war work intellectually challenging and satisfying, professionally the years were a waste. The Melbourne University Library refused him permission to borrow books, and there were so few mathematics books held by the Public Library that he read them all. At least by the end of the war he had made his decision to pursue a career in mathematics and not French.

Sydney 1946–47

Fortunately Eric's demobilization was hastened by T.G. Room, Professor of Pure Mathematics at the University of Sydney, who appointed Eric a Teaching Fellow in Mathematics in 1946. The tutoring and lecturing duties were very demanding, including a term at Armidale teaching five courses with a total of twelve lectures per week. He taught a third-year Honours course in Group Theory, a topic which he had never studied.

Eric applied for and was awarded the 'open' J.B. Watt scholarship and with encouragement from Professor Trendall applied for entry to Trinity College, Cambridge. At first unsuccessful, he was later accepted and departed for Cambridge in August 1947.

Cambridge 1947–53

Eric enrolled in the Cambridge Mathematical Tripos. Having a Sydney honours degree, he was permitted to take the three-year course in two years — a mixed blessing, since the solid two years of work for the crucial Part II examination had to be crammed into one. In the event, he passed all examinations with flying colours ('Wrangler' Part II, Distinction Part III). The academic atmosphere in Cambridge at this time was very stimulating for prospective number theorists. Professor L.J. Mordell's weekly Number Theory Seminar was attended by some twenty to thirty people, amongst whom there were a number of brilliant younger mathematicians. In 1949

Eric enrolled as a research student in the geometry of numbers with Mordell as supervisor. A fellow student, John Chalk, had first excited his interest in the subject by proposing a problem that was the starting point for his first paper, published in 1950.

What happened in these first years of research is vividly expressed in his own words: 'So in 1949 and 1950 I ate, drank and slept mathematics: reading and writing out notes on or translations of papers in the Cambridge Philosophical Society Library, working at problems on binary quadratic and bilinear forms and attending lecture courses and seminars'. His achievements over these two years were remarkable. By mid–1950 he had written seven papers for publication. These were submitted both for the Smith's Prize and for a Trinity College Fellowship. He was successful in both cases, the Smith's Prize being shared with another candidate. He was awarded his PhD (Cantab.) in 1952.

In 1951, Eric was appointed Assistant Lecturer in Mathematics, a three-year post in the first instance. In June that year he married Stewart Caird, an Australian, daughter of William and Emily Caird of Preston, Victoria. Their son, Peter, was born in Cambridge in 1953. In order to supplement the income from his lectureship, Eric undertook additional tutorial and examining work. Of his lectures at this time, Maurice Brearley (now Emeritus Professor of the University of Melbourne) writes:

> I was fortunate to attend a one-term course of lectures on linear algebra by Eric in Part 2 of the Cambridge Mathematical Tripos. His style was lucid and unhurried, his blackboard work always impeccable. He rarely consulted his notes during a lecture, giving the impression that he was not working from a planned script. Each lecture, however, ended precisely on time at a stage where there was a natural break in the mathematics; never was he part way through a proof when time ran out, which showed the whole had been meticulously planned. Eric had a dry sense of humour, far removed from any conscious joke. After introducing the concept of homomorphism he remarked: 'One of the Morph brothers'. The characteristic which I most appreciated was his ability to make even quite difficult concepts easy to grasp.

Despite his heavier teaching commitments, Eric's research continued to flourish. In particular, he continued a very fruitful collaboration already begun with Peter Swinnerton-Dyer.

Sydney 1953–58

With increasing family commitments and a barely adequate income from his fellowship and assistant lectureship, Eric was faced with serious decisions about his future. Cambridge was a great centre of number theory, in which he had established a firm reputation and made many friends. But he and Stewart had always intended to return to Australia. In 1953, he successfully applied for a Senior Lectureship at the University of Sydney. However, on being informed that he had been recommended for a full Lectureship in Cambridge, he turned the Senior Lectureship down. Not long after, an offer of a Readership at Sydney arrived, and this he accepted. Eric, Stewart and their infant son, Peter, left Cambridge in August 1953.

Before 1950, prospective research mathematicians from Australian universities needed to undertake further study overseas. By the early 1950s, however, the PhD degree was becoming established in mathematics at the University of Sydney and elsewhere. When Eric arrived, the Sydney department had a handful of research students who soon included his first PhD student, Jane Pitman. Later, he collaborated with algebraist Tim (G.E.) Wall on a significant joint paper. However, like most researchers in pure mathematics in Australia at the time, Eric had to work mainly in isolation, in marked contrast with his Cambridge experience.

Despite these seeming disadvantages, the years he spent in Sydney were some of the most fruitful of his career in terms of research. He also proved to be an accomplished lecturer at all levels, noted for his beautifully clear and well organized presentation.

On the personal side, Eric and Stewart were able to settle down as a family with reasonable financial security. Their daughter, Erica, was born in Sydney in 1956.

Eric's brilliant research in the 1950s soon gained recognition. In 1954, he was awarded the Edgeworth David Medal of the Royal Society of New South Wales. He was elected a Fellow of the newly founded Australian Academy of Science in the same year and was awarded its Thomas Rankin Lyle Medal in 1959.

Adelaide 1959–83

When H.W. Sanders retired as Professor of Mathematics at the University of Adelaide in 1958, the Council decided to replace him by two professors, one in Pure Mathematics and one in Applied Mathematics, who would alternate as Head of Department for periods of three years. Eric Barnes was appointed as Elder Professor of Pure Mathematics and moved to Adelaide with his family in January 1959. Ren Potts was appointed as Professor of Applied Mathematics and arrived a few months later.

Eric served initially as Head of the Department of Mathematics until the end of 1962. The position entailed a wide range of commitments outside the Department, including service on some key University committees, and his abilities soon gained respect. From 1963 onwards, while continuing his mathematical commitments, Eric became increasingly involved in University administration. It was a loss to mathematics when he vacated the Chair of Pure Mathematics to become one of two Deputy Vice-Chancellors in 1975. As Deputy Vice-Chancellor, he had major responsibility in several areas, including academic matters and University entrance, and chaired the University's Co-ordinating Committee and the Committee of Deans. In 1980, restructuring of the University's management saw the two Deputy Vice-Chancellor positions discontinued after the end of their first term, and in 1981 the two incumbents returned to their respective departments as Professors. Eric was warmly welcomed by what was then the Department of Pure Mathematics and served as elected Chairman in 1982. In May 1983, he took up the opportunity of early retirement.

The Mathematics Department

In 1959, despite the presence of some very able researchers, the Mathematics Department was relatively inactive. It had produced only one PhD and very few honours graduates and its main focus was on service teaching. On the arrival of Ren Potts the two new professors began a friendly co-operation that helped to transform the Department into an active modern department with high standards in both teaching and research.

The structure of the mathematics courses was streamlined by introducing General Mathematics, abolishing Applied Mathematics I, and replacing Pure Mathematics I by Mathematics I. Priority was given to establishing a comprehensive fourth-year honours course, with flexible prerequisites. Honours projects were introduced with the allocation of staff as honours supervisors. The results were dramatic. For the first time, Honours Mathematics became an attractive option for mathematically inclined students. The number of honours students increased from one in 1959 to twenty in 1964. The Australian Mathematical Society coincidentally started collecting and publishing from 1959 the yearly number of honours students graduating in the mathematical sciences in the universities in Australia. Adelaide soon topped the list, above much larger universities.

The establishment of a strong postgraduate program was of course more protracted. A staff member graduated with a PhD in 1961, two students in 1964, three in 1966, four in 1967 and seven in 1968. In the yearly lists mentioned above, Adelaide was soon near the top.

A separate Department of Computing Science was already established and in 1968 Statistics separated from Mathematics to form its own department. In 1970 the applied mathematicians proposed a further split into Pure and Applied Mathematics. Although Eric was not in favour, he did not oppose it and years later he wrote that the move was 'completely vindicated'. The separation took effect in 1971, with Eric and Ren as the Heads of the two new departments which continued to co-operate closely and shared some staff. The new structure helped provide a firm foundation for the establishment of the Faculty of Mathematical Sciences for which Eric was to prove the strongest advocate and its first Dean in 1973. The Faculty comprised five departments, Pure, Applied, Statistics, Computing Science and (also in Science) Mathematical Physics. The new Faculty was by student numbers the third largest in the University (behind Arts and Science). Its new degree provided the flexibility required by the growth of the mathematical sciences and their links with many other areas.

Between 1959 and 1974, Eric gave effective academic leadership in all aspects of the work in pure mathematics, including teaching, research, supervision of research and honours students, and on-going curriculum development.

The University

Eric gave extensive and dedicated service to the University in terms of administra-

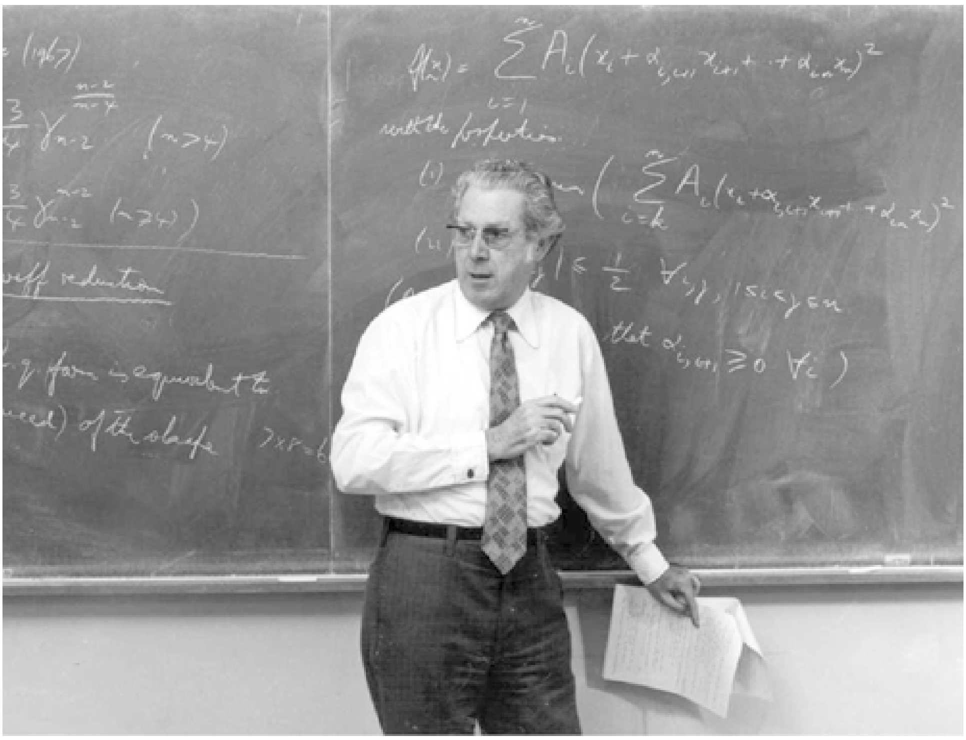

Figure 1. Professor Barnes lecturing on geometry of numbers in 1982.

tion, committee service and service on behalf of the University to outside bodies. Apart from his terms as Head of Department, Faculty Dean and Deputy Vice-Chancellor, his service included terms as Chief Examiner in Mathematics for the Public Examinations Board (for many years), Chairman of the Board itself, Chairman of the Board of Research Studies, Chairman of the Education Committee (now the Academic Board), and elected member of the University Council.

Professional service

During his time in Adelaide, Eric was active in the wider mathematical and scientific community.

Through his early work as Chief Examiner in Mathematics and the associated chairmanship of the Mathematics Syllabus Committee, he soon established friendly relations with local school mathematics teachers and went on to play a leading role in connection with school mathematics. With the support of A.W. Jones of the State Education Department, Eric was responsible for establishing the Mathematical Association of South Australia (the state professional association of mathematics teachers) and became its foundation President (1959–61). He gave practical help to teachers through presentations at conferences of the Association and especially by writing a useful textbook (see (47) in the Bibliography below) jointly with mathematics teacher Bruce Robson.

Eric was a foundation member of the Australian Mathematical Society (from 1956) and served the Society in a number of roles, including President (1962–64), Council Member, Director of the Summer Research Institute (1962), and member of the Editorial Board, and later Associate Editor (1967–1974), of the Society's Journal. While in Adelaide he also served the Australian Academy of Science as Council Member (1962–64) and Secretary, Physical Sciences (1972–76).

Teaching

During the early 1960s, both the new professors taught mathematics at first- and second-year levels. This helped to establish the curriculum and attract the interest of students. Eric's meticulous lecture notes were extremely useful to the lecturers who took over subjects he had taught and set a high standard of preparation.

The outstanding quality of Eric's teaching was immediately recognised. Both in the early 1960s and in 1982, students particularly appreciated his teaching of second-year Pure Mathematics. Eugene Seneta (now Professor of Mathematical Statistics at the University of Sydney) writes[1] (see References below) that 'he heavily influenced the direction of my future work through his lucid teaching of Pure Mathematics 2 in 1961'.

Much of Eric's teaching was at third-year and honours levels, and his honours courses attracted graduate students and staff members as well as honours students. While most of his honours courses were on topics in number theory, including the geometry of numbers, there were occasional exceptions. A notable example was his 1962 course on Linear Inequalities, mentioned by several of those who attended, including Seneta. Eric's interest in this then new topic had been stimulated by A.W. Tucker on a visit to Sydney, and it was highly relevant to his own research. The course was an eye-opener and formed the basis of extensive and continuing research on mathematical programming by some of the more applied students and staff.

Research and scholarly work

In the decade 1962–72, Eric supervised five successful PhD students, Tom Dickson, Paul Scott, Rod Worley, Peter Blanksby (for two years) and Dennis Trenerry. He assisted in supervising the MSc work of staff member Marta Sved, who went on to further research and a PhD, and also supervised the MSc work of the late Christopher

Nelson, who submitted his one paper[2] in 1972, the year before, sadly, he was drowned in floods in New South Wales. All six of Eric's PhD students proceeded to careers in university mathematics and to further research, and three, Pitman, Scott and Blanksby, joined the Adelaide department. He gave his research students significant and challenging research problems, and their publications bear witness to his influence. Paul Scott (who recently retired as Associate Professor at the University of Adelaide) writes that 'Eric's main legacy to me was the ability to write clear and concise mathematics' and 'I feel privileged to have had Eric first as my mentor, and later as a colleague'.

As might be expected in view of his many other commitments, Eric's own research output slowed markedly after he took up the Chair in Adelaide. However, during the next twelve years or so, his work gained increasing recognition, and papers in the geometry of numbers by his students (some joint with Eric) appeared at a steady rate. Leading international number theorists visited Adelaide, and Eric's involvement ensured that the newly established *Journal of the Australian Mathematical Society* published papers in number theory by both international and local authors. By the early 1970s, Adelaide was among the half dozen internationally recognised centres of progress in the geometry of numbers, of which the largest was the Moscow/Leningrad school centred on the Steklov Institute in which B.N. Delone and S.S. Ryškov were leading figures.

An Australian Research Grants Committee award enabled Eric to appoint a Postdoctoral Fellow for 1974–76 for a project related to his recent research and to correspondence with Ryškov. The appointee was Michael Cohn (former research student of Paul Scott and of C.A. Rogers, London), and the project was very successful.

While at Adelaide, Eric had three periods of overseas leave. In 1965, he and his family spent a year of study leave in Kuala Lumpur and Bangkok, part of the aim being to further mathematics education in the region. In 1975 he had a two-month overseas study tour as Deputy Vice-Chancellor, and in 1981 he had a full year of study leave. Early in 1981 he met Neil Sloane (then of Bell Telephone Laboratories, New Jersey), with whom he began research collaboration. A visit to the US and Canada later in the year provided further opportunity for work with Sloane.

Personal

Eric Barnes was an exceptionally quick and incisive thinker with an excellent memory. He devoted time and hard work to any matter which he took up and had a remarkable ability to master complex detail and identify the essentials. He was also a gifted expositor whose presentations were clear, logical and appropriate to their audiences. These qualities underlay his mathematical research and teaching and contributed greatly to his work in administration and academic management.

Eric's intellectual abilities could at times be daunting, and by nature he had a low tolerance for inaccuracy. In debate on large University committees, these characteristics sometimes led to an abrasive style of argument.

Eric gained and retained the respect and affection of his colleagues in pure mathematics. While he could argue cogently for his own point of view at departmental meetings, he was not dominating and was very much the opposite of a 'God-Professor'. He had a genuine concern for students at all levels and took a special interest in students from overseas, mathematically gifted students, and those who did not quite fit the system. His colleagues found him approachable, supportive and encouraging, particularly to new arrivals, those with less experience, and new researchers.

Among Eric's recreational interests were music, bridge, chess, reading and a love of language and words. He could lighten the atmosphere with a witty turn of phrase and he remained fluent in idiomatic French. He was a competent pianist and those who were there remember a happy Mathematics Department party at the home of the Potts family, around 1963, when musical entertainment was provided by a trio (Eric Barnes, piano, George Szekeres, viola, Ren Potts, clarinet) and novice pianist Maurice Brearley.

In 1984 the E.S. Barnes Prize (for third-year Pure Mathematics) was established, thanks to former students, friends and colleagues, in recognition of his contribution to pure mathematics.

Later Years

From the 1960s, Eric had to cope with some health problems, in particular, a chronic respiratory condition. By the early 1980s these problems had increased and this was a factor in his early retirement. After retirement his health gradually deteriorated, but he maintained contact with the Pure Mathematics Department as an Honorary Visiting Research Fellow and participated actively in the weekly Number Theory Seminar until 1992.

In his last few years Eric was house-bound and had occasional visits to hospital. He died on 16 October 2000.

Mathematical Work

Apart from his admirable note (41) (see Bibliography below) and his contributions to the books (47), (48) and (49), Barnes's mathematical publications belong to the geometry of numbers. We start with some necessary background and then indicate the main themes of his work.

Background

Classical problems in number theory include not only Diophantine equations but also Diophantine inequalities. A typical *homogeneous* Diophantine inequality of degree 2 is

$$|x^2 + xy + y^2| \leq 1,$$

where we seek non-null integral solutions, that is, solutions $x = u$, $y = v$ such that u and v are integers (0, ± 1, ± 2,…), not both zero. There are also corresponding *inhomogeneous* inequalities involving a polynomial in x and y whose terms are not all of the same degree.

Consider a plane with a standard rectangular Cartesian co-ordinate system. The 'integral lattice', which consists of all points whose co-ordinates are integers, is a particular example of the concept of 'lattice'. Investigation of a problem on Diophantine inequalities in two variables is often helped by consideration of an equivalent geometrical problem on lattice points in a region of the plane.

More general Diophantine inequalities involve *forms* of degree k in n variables. These are homogeneous polynomials of degree k with real coefficients, in n real variables. Such a form is *binary* if $n = 2$, *ternary* if $n = 3$. A form is *quadratic* if it has degree 2, *integral* if its coefficients are integers, *indefinite* if it takes both positive and negative values, and *positive*, or positive definite, if its value is always positive except when all variables are zero.

The geometry of numbers is a major branch of number theory that was introduced by Minkowski in the 1890s. The subject deals with n-dimensional lattices and their relationship to bodies in n-dimensional space for all $n \geq 2$. It provides an effective geometrical framework for many problems on Diophantine inequalities and has important applications both within and beyond number theory.

Main themes

Barnes's research publications can be conveniently grouped under three headings:

Indefinite forms — Part 1,
Indefinite forms — Part 2,
Positive quadratic forms and lattices.

The work on indefinite forms began in Cambridge, continued in Australia, mainly during the 1950s, and provided topics for the theses of Pitman, Worley and Blanksby. Part 1 of this work deals with *homogeneous Diophantine inequalities for indefinite forms* and uses purely arithmetic methods. Part 2 deals with *inhomogeneous inequalities for indefinite binary quadratic forms* and related topics, and uses two-dimensional lattices.

From the mid-1950s onwards, Barnes's major research interest was in *positive quadratic forms in n variables* and *n-dimensional lattices*. This theme provided topics for the theses of Scott, Dickson, Trenerry and Nelson, and also for the major research project with Cohn. The problems Barnes considered include both homogeneous and inhomogeneous inequalities for positive forms. These are equivalent to problems on *packing and covering of n-dimensional space with equal spheres* whose centres are at the points of a lattice.

Barnes was interested in solving specific problems rather than in developing abstract theory, and his preferred mathematical tools were those of discrete mathematics and geometry. The research problems in the geometry of numbers which he addressed fitted well with these tastes, gave scope for all his intellectual abilities and mathematical powers, and required, in addition, a high degree of creative insight. The results he obtained are almost all in some sense best possible, and all his papers bear the hallmarks of his style: clarity, economy and beautiful organization.

We shall discuss Barnes's work on each of the three main themes further below. The papers considered in more detail have been selected to reflect the range of his main work. We give some background to place the work in context.

References

References (1), (2), etc., are to the Bibliography of Barnes's publications at the end of this article. References such as Cassels[3] are to the list of References immediately before the Bibliography. Standard references for the geometry of numbers are Cassels[3] and Gruber-Lekkerkerker[4]. References such as GL, 17, or GL, xi, are to sections (17 or xi) of Gruber-Lekkerkerker[4].

Indefinite Forms — Part 1

Background on indefinite binary quadratic forms

We denote by R^2 the space of all *real* vectors $\mathbf{x} = (x, y)$ (viewed as points of the plane) and by Z^2 the lattice of all *integral* points $\mathbf{u} = (u, v)$. Consider a binary quadratic form

$$f = f(x, y) = ax^2 + bxy + cy^2$$

with real coefficients a, b, c and discriminant $d(f) = b^2 - 4ac \neq 0$. We investigate the values $f(\mathbf{u})$ for $\mathbf{u} = (u, v)$ in Z^2, and particularly the *homogeneous minimum*

$$M(f) = \inf |f(u, v)| \; (u, v \text{ integers, not both } 0),$$

where 'inf' means infimum, or greatest lower bound.

A linear transformation T on R^2 maps Z^2 onto itself if and only if its 2×2 matrix has integral entries and determinant ± 1. We shall call such a transformation T an *automorph* of Z^2. A binary quadratic form g is *equivalent* to f if $g(x, y)$ is identical with $f[T(x, y)]$ for some automorph T of Z^2. In this case, $d(g) = d(f)$, the set of all values $g(\mathbf{u})$ with $\mathbf{u}$ in Z^2 coincides with the set of all $f(\mathbf{u})$ with $\mathbf{u}$ in Z^2, and $M(g) = M(f)$. The value of $M(f) / |d(f)|^{1/2}$ is unchanged if f is replaced by a form equivalent to a non-zero multiple of f. The concepts of homogeneous minimum and equivalence are easily extended to any form in any number of variables.

For a form f as above with discriminant $d(f)=d$ and homogeneous minimum $M(f)=M$, suppose that f is indefinite ($d>0$). Then

$$M \leq 5^{-1/2}\, d^{1/2},$$

with equality if and only if f is equivalent to a multiple of $F_0 = x^2 - xy - y^2$. However, if equality does not hold, then

$$M \leq 8^{-1/2}\, d^{1/2},$$

so that the constant $5^{-1/2}$ is 'isolated'. This first example of isolation was discovered by Korkine and Zolotareff in 1873, and led to Markoff's major study published in 1879–80. Markoff found a remarkable infinite sequence of forms $F_0, F_1, F_2, \ldots$ (now the *Markoff forms*). The strictly decreasing sequence of values $Md^{-1/2}$ for these forms starts with $5^{-1/2}$, $8^{-1/2}$, $5(221)^{-1/2}$,… and converges to 1/3. Markoff's main result is that M is at most $d^{1/2}/3$ unless f is equivalent to a multiple of some Markoff form.

Much more is now known about the structure of the set of all possible values of $Md^{-1/2}$, which provides a standard of comparison for the 'spectra' of constants arising in other problems. (The reciprocals $d^{1/2}\, M^{-1}$ form the 'Markoff spectrum'.)

For the work above, Markoff used an early version of the now classical *continued fraction method*. Since much of Barnes's work on indefinite forms relies on this method, we indicate the main ideas.

Suppose again that $f = ax^2 + bxy + cy^2$ is indefinite with $d(f) = d > 0$ and $M(f) = M$. The equation $ax^2 + bx + c = 0$ has real solutions θ, Φ with $|\theta| \leq |\Phi|$, called the *roots* of f. We assume that $a \neq 0$ and the roots are irrational (otherwise $M = 0$). Starting from a suitable equivalent form f_0 with roots θ_0, Φ_0, we obtain a two-way infinite sequence of positive integers

$$\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$$

from the *regular* (or 'ordinary') *continued fractions*

$$-\Phi_0 = (a_1;\, a_2\, a_3, \ldots) = a_1 + \frac{1}{a_2 + 1/(a_3 + \ldots)},$$

$$1/\theta_0 = (a_0;\, a_{-1}, a_{-2}, \ldots).$$

The a_i determine a corresponding sequence of equivalent 'reduced' forms f_i (these are unrelated to the Markoff forms F_i).

If u, v are integers, not both zero, such that $|f(u, v)| < d^{1/2}/2$ then $f(u, v)$ is equal to $f_i(1, 0)$ for some i. Since there is a simple formula for $f_i(1, 0)$ in terms of continued fractions, this gives us a powerful tool for studying these values $f(u, v)$.

Restricted homogeneous minima of indefinite forms

The papers (2) to (7) cover Barnes's substantial early research in Cambridge [apart from (1)]. The interrelated papers (2) to (6) deal with 'restricted' minima of certain indefinite forms, and the same circle of ideas includes (7) and the later paper (17). All of this work depends on masterly use of the continued fraction method and sequences of reduced forms, for relevant binary quadratic forms.

The *restricted homogeneous minimum* of a form $f(x, y, z, t)$ in four variables is the infimum of the values of $|f(x, y, z, t)|$ at integral values of the variables satisfying $xt - yz = \pm 1$. Within this context, we denote this restricted minimum by $M(f)$. Davenport and Heilbronn had studied $M = M(f)$ for $f = (ax + by)(cz + dt)$ where $\Delta = ad - bc \neq 0$. (See last paragraph of GL 43.2.) In contrast to Markoff's results for binary forms, they found that the third largest value of $M\,\Delta^{-1}$ is not isolated.

In (2) to (4) Barnes considered the form $f(x, y, z, t)$ given by $f = q(x, y)q(z, t)$ where q is an indefinite binary quadratic form with discriminant $d=d(q)$. In (2), he studied the values of Md^{-1}, where $M = M(f)$. He found a sequence of quadratic forms Q_{-1}, Q_0, Q_1, Q_3, Q_5,… and demonstrated a remarkable analogue of the Markoff phenomenon for the forms $h_i = Q_i(x, y)Q_i(z, t)$: The values of Md^{-1} for the h_i form a strictly decreasing sequence converging to a specified limit C and Md^{-1} is at most C except when q is equivalent to a multiple of some Q_i. The papers (3), (4) deal with restricted minima associated with 'asymmetric' and 'one-sided' inequalities for f as above. They include further instances of Markoff phenomena, some of which contrast with known results for binary forms.

In (5), Barnes proved the theorem of Davenport and Heilbronn mentioned above by the methods of (2). This led to his study in (6) of best possible upper bounds for the restricted homogeneous minimum $M(f)$ of a more general 'bilinear' form $f(x, y, z, t)$.

One-sided inequalities for indefinite quadratic forms

Three 1955 papers on indefinite quadratic forms stemmed from a visit to Sydney by A. Oppenheim. One, (17), was mentioned above. Before discussing the others, we need some further vocabulary.

The *matrix* of a ternary quadratic form $f(x_1, x_2, x_3)$ is the 3×3 symmetric matrix $A = [a_{ij}]$ such that

$$f(\mathbf{x}) = f(x_1, x_2, x_3) = \sum a_{ij}\, x_i\, x_j\,,$$

with summation over all i, j such that $1 \le i \le 3$, $1 \le j \le 3$. The *determinant* of f is det$A$ (the determinant of A). If this is non-zero then f can be expressed as $\pm L_1^2 \pm L_2^2 \pm L_3^2$, where the L_i are linear forms, and f is said to be of *type* (r, s) when there are r plus signs and s $(=3-r)$ minus signs. If f is indefinite, its *non-negative minimum* $M^+(f)$ is the infimum of the non-negative values taken by f at integral values of the variables, not all zero, and its *positive minimum* is defined similarly. (These may differ because there may be non-null integral solutions of $f(\mathbf{x})=0$). These concepts extend to quadratic forms in n variables for any n.

In 1953 Oppenheim had given best possible upper bounds for the positive minima for $n = 3, 4$ (see GL, 44.4). In (18), Barnes addressed the more difficult corresponding problem for $M^+(f)$, and this work was further extended in the joint paper (20) with Oppenheim. For $n \ge 2$, $r \ge 1$ and $s \ge 1$, we consider all indefinite quadratic forms f in n variables of type (r, s) with non-zero determinant D. The constant $\kappa_{r,s}$ is the least positive constant such that for all such forms f we have

$$M^+(f) \le (\kappa_{r,s}\,|D|)^{1/n}.$$

The cases $n = 2, 3, 4$ are the only ones of interest since, thanks to the 1987 breakthrough of Margulis (see Dani and Margulis[5]) we now know that the positive minimum of f is zero if $n \ge 5$.

For $n = 2$, it was known (see GL, xiv.4) that $\kappa_{1,1}$ is 4 and is not isolated. In (18), Barnes obtained a key lemma on positive values of indefinite binary quadratic forms by the continued fraction method. Using this, he showed that $\kappa_{2,1}$ is 4/3 and is isolated, and derived upper bounds in the other cases with $n = 3, 4$. The co-operation with Oppenheim yielded the values of $\kappa_{1,2}$ and $\kappa_{2,2}$.

Later, in his thesis, Worley studied indefinite ternary quadratic forms and achieved major progress on asymmetric and one-sided inequalities (see Worley[6, 7]).

The work on $\kappa_{r,s}$ was carried further by Worley[8], and by Jackson, who, in particular, evaluated $\kappa_{3,1}$ (see GL, xiv, for references). It seems that $\kappa_{1,3}$ has still not been precisely determined.

As well as being of independent interest, results on one-sided and asymmetric inequalities often play an important role as auxiliary results in other problems. For example, the value of $\kappa_{2,1}$ was essential to Barnes's later work in (22) and was used similarly by Raka[9].

Indefinite Forms — Part 2

Let $f = f(x, y)$ be an indefinite binary quadratic form with discriminant $d(f) > 0$. For each real $\boldsymbol{\alpha} = (\alpha, \beta)$, let

$m(f; \alpha, \beta) = \inf|f(u+\alpha, v+\beta)|$ (u, v integers),

and write

$$m(f; \alpha, \beta) = m(f, \boldsymbol{\alpha}).$$

If $(\alpha', \beta') \equiv (\alpha, \beta)$ (mod 1), that is $\alpha' - \alpha$ and $\beta' - \beta$ are integers, then $m(f; \alpha', \beta')$ is equal to $m(f; \alpha, \beta)$. The *inhomogeneous minimum* $m(f)$ is the supremum, or least upper bound, of the values of $m(f; \alpha, \beta)$ for all real α, β:

$$m(f) = \sup m(f; \alpha, \beta) \ (\alpha, \beta \text{ real}).$$

The behaviour of $m(f)$ under equivalence of forms is exactly similar to that of the homogeneous minimum $M(f)$.

The inhomogeneous minimum of any form in any number of variables is defined similarly. The concept arises naturally in algebraic number theory. For quadratic forms it can be interpreted in terms of lattice coverings of Euclidean space.

We now look at the contributions of Barnes related to inhomogeneous minima of indefinite binary quadratic forms.

Two-dimensional lattices and Minkowski's theorem

Let $s_1x + t_1y$ and $s_2x + t_2y$ be two real linear forms whose coefficient matrix

$$L = \begin{bmatrix} s_1 & t_1 \\ s_2 & t_2 \end{bmatrix}$$

has non-zero determinant. The product

$$f = (s_1x + t_1y)(s_2x + t_2y)$$

is an indefinite binary quadratic form with discriminant $(\det L)^2$. Every indefinite binary quadratic form can be factorised in this way and the linear factors are unique up to suitable multiples.

The *two-dimensional lattice* determined by the two given linear forms, or equivalently by the matrix L, consists of all points $\mathbf{w} = (w_1, w_2)$ such that for some *integral u* and v we have

$$w_1 = s_1u + t_1v,$$

$$w_2 = s_2u + t_2v.$$

(The vectors $\mathbf{s} = (s_1, s_2)$, $\mathbf{t} = (t_1, t_2)$ form a 'basis' of Λ.) If Λ is also determined by a matrix L' then $|\det L'| = |\det L|$, and so the *determinant* of Λ is uniquely defined as $\det \Lambda = |\det L|$.

Minkowski proved the following seminal theorem geometrically in the 1890s. Let Λ be the lattice determined by two real linear forms $L_1(x, y)$, $L_2(x, y)$ with non-zero determinant. Then for each point $\gamma = (\gamma_1, \gamma_2)$ in the plane there is a point $\mathbf{w} = (w_1, w_2)$ of Λ such that

$$|(w_1 + \gamma_1)(w_2 + \gamma_2)| \le \tfrac{1}{4}\det \Lambda.$$

Equivalently, in terms of forms: Let f $(=L_1L_2)$ be an indefinite binary quadratic form with discriminant $d(f) = d$. Then for each real $\boldsymbol{\alpha} = (\alpha, \beta)$ there is an integral vector $\mathbf{u}$ such that

$$|f(\mathbf{u} + \boldsymbol{\alpha})| \le \tfrac{1}{4}d^{1/2}.$$

Thus $m(f, \boldsymbol{\alpha})$ is at most $d^{1/2}/4$ for all real $\boldsymbol{\alpha}$ and the inhomogeneous minimum $m = m(f)$ is at most $d^{1/2}/4$. *Minkowski's constant* 1/4 is best possible but is not isolated.

Interest in improving Minkowski's theorem led to detailed study of particular forms f and investigation of alternative upper bounds for $m(f)$. Heinhold in 1939 and others studied the 'principal norm forms' of real quadratic fields. (See GL, 47.5.) These are of special interest because their inhomogeneous minimum is connected with the question of whether or not the field is *Euclidean*, that is, its domain of 'integers' has a 'Euclidean algorithm' (and hence has the unique factorisation property).

In (1), Barnes obtained an upper bound which is often sharper than Minkowski's: If $f = ax^2 + bxy + cy^2$ is indefinite with inhomogeneous minimum $m = m(f)$ then

$$m \le \tfrac{1}{4}\max\{|a|, |c|, \min|a \pm b + c|\}.$$

He gave some significant applications, including simple verification of most known examples of Euclidean real quadratic fields. Related work is discussed in GL, 47.4.

The automorph method

While in Cambridge, Barnes pursued the theme of (1) further in major collaborative research with Swinnerton-Dyer. They used two different approaches, both highly geometrical, and from then onwards Barnes's work was underpinned by geometrical ideas, even when the details were arithmetical.

In the joint papers (8),(9) they studied the inhomogeneous minima of norm forms of real quadratic fields. For square-free integral $k > 1$, the field $Q(\sqrt{k})$ is obtained by adjoining $\sqrt{k}$ to the field Q of rational numbers and its *principal norm form* is

$$g_k = x^2 + xy - \tfrac{1}{4}(k-1)y^2$$

if $k - 1$ is divisible by 4, and, otherwise,

$$g_k = x^2 - ky^2.$$

For this purpose, Barnes and Swinnerton-Dyer developed what we shall call the *automorph method*, a general geometric method applicable to *integral forms* (or multiples thereof).

Consider an integral indefinite form $f = ax^2+bxy+cy^2$ with $a \neq 0$, irrational roots, and inhomogeneous minimum $m = m(f)$. An *automorph* of f is an automorph T of Z^2 such that the form $f\,[T(x, y)]$ is identical with $\pm f(x, y)$. The integral form f has automorphs T of infinite order, and the method is based on the fact that, for such T and real $\boldsymbol{\alpha} = (\alpha, \beta)$, we have

$$m[f; T(\boldsymbol{\alpha})] = m(f; \boldsymbol{\alpha}).$$

An important role is played by the orbit (mod 1) of $\boldsymbol{\alpha}$ under T for certain 'exceptional' points $\boldsymbol{\alpha}$. In (8), Barnes and Swinnerton-Dyer obtained a group of theorems which together provide a firm theoretical basis for finding m by this approach. In (8) and (9) they also gave some general theorems on $m(f; \boldsymbol{\alpha})$ for given integral f. In particular, the set of all values of $m(f; \boldsymbol{\alpha})$ is closed.

For real f, let $m_2 = m_2(f)$ be the supremum of the values of $m(f; \boldsymbol{\alpha})$ which are strictly less than $m = m(f)$. Clearly $m_2 \leq m$. If $m_2 < m$ then the minimum m is isolated and m_2 is the *second* (*inhomogeneous*) *minimum* of f. If m_2 is isolated there will be a third minimum $m_3 < m_2$, and so on. As conjectured in (9) and later shown by Godwin,[10] the second minimum m_2 of f need not be isolated even if f is integral.

Barnes and Swinnerton-Dyer used the automorph method to study $m = m(f)$ for the norm form $f = g_k$ in many cases, including all square-free $k \leq 101$. In (8) they presented theorems covering different possibilities and tabulated their results, many of which were new, with references for known results. In all but a handful of cases, they evaluated m and found the points at which $m(f; \boldsymbol{\alpha})$ takes this value, and often they also evaluated m_2. Later, by a modification of their method, Godwin[11] filled the remaining gaps for $k \leq 101$.

In (9), Barnes and Swinnerton-Dyer extended the automorph method to deal with an infinite sequence of minima $m = m(f), m_2, m_3, \ldots$ and studied the norm forms $f = g_k$ for $k = 11, 13$ in detail. For $k = 11$, they found an analogue of the Markoff phenomenon, and their results are similar to those obtained earlier by Davenport for $k = 5$ and Varnavides for $k = 2$ (see GL, end of 47.5, for references). The norm form f for $k = 13$ has unusual properties, and the results for this case are surprisingly complicated.

Barnes and Swinnerton-Dyer conjectured in (9) that for all integral indefinite binary quadratic f the inhomogeneous minimum m is always rational, isolated, and taken by $m(f; \boldsymbol{\alpha})$ at some *rational* point $\boldsymbol{\alpha} = (\alpha, \beta)$ with α, β both rational. See Berend and Moran[12] for progress towards this.

Euclidean real quadratic fields

For square-free $k > 1$, the real quadratic field $Q(\sqrt{k})$ with norm form $f = g_k$ is Euclidean if and only if $m(f; \boldsymbol{\alpha}) < 1$ for all *rational* points $\boldsymbol{\alpha} = (\alpha, \beta)$. A 1951 theorem of Davenport led via this criterion to completion of the proof that there are no further real Euclidean quadratic fields beyond those on the 'classical' list of 17 Euclidean fields, all with $k < 101$. (See Ennola[13] and GL 48.3.) The results of (8) (9) and Godwin[11] confirmed these conclusions for all square-free $k \leq 101$, with one exception. Barnes and Swinnerton-Dyer showed in (8) that the last field on the list, $Q(\sqrt{97})$, had been wrongly listed as Euclidean. In 1958 Ennola[13] improved the Davenport result and drew on ideas from several articles [in particular (1) and (8)] to give the first unified proof of the correctness of the final list of 16 values of k for which $Q(\sqrt{k})$ is Euclidean. Barnes made further contributions related to this topic in the later papers (31) and (39).

Background on Delone's geometric divided cell algorithm

The second method of Barnes and Swinnerton-Dyer is the *divided cell method*, based on a geometric algorithm for inhomogeneous problems introduced by Delone[14] in 1947. (See GL, 48.1 for a detailed account. The automorph method and the significant further developments discussed below are not covered in GL.)

Delone's algorithm is formulated in terms of grids. A *grid* Γ associated with a two-dimensional lattice Λ is a translate of Λ, $\Gamma = \gamma + \Lambda$, for some specified real point $\gamma = (\gamma_1, \gamma_2)$. The points of Γ and the lines they determine are called grid points and grid lines, and the *determinant* of Γ is $\det\Lambda$. A parallelogram P whose four vertices are grid points is a *cell* of Γ if there are no other grid points inside P or on its boundary, or, equivalently, if P has area $\det \Gamma$. A cell is *divided* if, further, it has one vertex in each of the four quadrants determined by the axes $x = 0$, $y = 0$.

Suppose a grid Γ has no grid point on either of the axes and no grid line parallel to an axis. Delone showed that Γ has at least one divided cell, C_0, say, and gave a simple geometric algorithm which starts from C_0 and yields a two-way infinite chain of divided cells which includes all divided cells of Γ:

$$\ldots, C_{-2}, C_{-1}, C_0, C_1, C_2, \ldots$$

He obtained basic results on divided cells and gave some significant applications.

Inhomogeneous problems on indefinite binary quadratic forms are equivalent to problems on the value of xy at grid points (x, y) of an appropriate grid Γ. The divided cell algorithm is important because in order to determine the infimum of $|xy|$ for all grid points, or for all grid points in a certain quadrant, it is sufficient to consider grid points which are vertices of divided cells.

The divided cell method for asymmetric inequalities

For real $\tau \geq 1$, consider grids Γ with no grid point in the asymmetric region $-1 < xy < \tau$, and let $D(\tau)$ be the infimum of their determinants. In (15) Barnes and Swinnerton-Dyer extended the divided cell method to study $D(\tau)$ (see GL, 50.1, regarding related work). They first gave a full presentation of the basic theory and obtained formulas for the vertices of the divided cells in terms of two sequences of integers a_i and ε_i which arise from Delone's construction. The results involve 'semi-regular' continued fractions of a special type, together with series expansions.

A grid is *symmetric* if one of its cells has the origin (0, 0) as its centre. Such grids have all $\varepsilon_i = 0$ and are easier to handle. Barnes and Swinnerton-Dyer proved that symmetric grids are sufficient for evaluation of $D(\tau)$ and extended the theory further for this type of grid. They obtained lower bounds for $D(\tau)$ and illustrated both the power of the method and the complex nature of $D(\tau)$ by precise evaluation of $D(\tau)$ for τ near 2. In 1991, Grover and Raka[15] revisited this work, filled a gap in (15), and used the method for further detailed study of $D(\tau)$.

Davenport had used results on inhomogeneous asymmetric inequalities for binary forms to obtain the analogue of Minkowski's constant for an indefinite *ternary* quadratic form $f(x, y, z)$ with non-zero determinant (see GL, 49.4). In the major papers (14) and (22), Barnes used the divided cell method for asymmetric inequalities to carry this work much further. In (32), without using (15), he also gave another proof of a result of Blaney on one-sided inhomogeneous inequalities for indefinite ternary quadratic forms (see GL, 50.3). In 1993, Raka[9] obtained a major extension of this result by using some of the work of Grover and Raka mentioned above, together with other auxiliary results.

The divided cell method for inhomogeneous minima of forms

Let $f(x, y)$ be a real indefinite binary quadratic form with irrational roots and inhomogeneous minimum $m = m(f)$. The evaluation of m is equivalent to a problem on vertices of divided cells in terms of associated sequences of integers a_i and ε_i. However, this is more difficult than the problem in (15) because infinitely many different sequences of a_i must be considered and the ε_i are in general non-zero. In the important paper (16), Barnes overcame these difficulties by deriving further theoretical results on the divided cell method. He thus developed the method as an arithmetical tool for the study of $m(f)$ for real forms f as above. He illustrated its advantages over the automorph method by applying it to the relatively difficult norm forms $f = g_k$ with $k = 19$ and $k = 46$ (correcting (8) for $k = 19$).

In (21) Barnes modified the divided cell method to deal with the inhomogeneous Diophantine approximation constants $k(\theta, \beta)$, $k^+(\theta, \beta)$, for positive irrational θ and suitable real β. These constants are related to Diophantine inequalities of the type

$$|x(\theta x - y - \beta)| < C$$

with the conditions $x \neq 0$ or $x > 0$. As easy applications, Barnes gave short proofs of two parallel theorems on these constants. The first strengthened a much earlier result of Morimoto (Fukasawa)[16] (whose other work seems to have been neglected in the literature).

In their theses Pitman and Blanksby gave further major applications of the divided cell method and supplemented the theory (see Pitman[17, 18], Blanksby[19, 20]).

Recent developments

In the decade from 1973, research on inequalities for indefinite binary quadratic forms concentrated mainly on homogeneous problems involving the Markoff spectrum and related topics. Since about 1990 there has been renewed interest in the inhomogeneous problems investigated by Barnes, with emphasis on the whole spectrum of values of $m(f; \boldsymbol{\alpha})$ or $k(\theta, \beta)$ or $k^+(\theta, \beta)$ for fixed f or θ. In particular, William Moran (who succeeded Barnes in the Chair of Pure Mathematics at the University of Adelaide) and his collaborators have contributed in this area. The divided cell method remains a powerful tool, and investigation of new approaches to the automorph method and the ideas of (8), (9) and (15) has begun. Berend and Moran[12] used the methods of topological dynamics to study the values of $m(f; \boldsymbol{\alpha})$ for indefinite binary quadratic f. Recent developments are further illustrated by the work of Grover and Raka[15] and the paper of Raka[9] mentioned earlier, and by the papers of Cusick, Moran and Pollington[21] and of Pinner[22] on inhomogeneous approximation.

Positive Quadratic Forms and Lattices

From the mid-fifties on, Barnes's major research interest was in *positive* quadratic forms in n variables and their associated *lattices* of points in n-dimensional space. The standard reference work in the area is Conway-Sloane.[23]

Mathematical background

For $n \geq 3$, n-dimensional lattices are a natural extension of the case $n = 2$ discussed earlier. We give an informal account of the geometry of 3-dimensional lattices. In the process, some of the basic ideas of the general n-dimensional theory will be introduced.

Consider 3-dimensional space as in Euclidean geometry with one point O permanently selected as origin, and let A,B,C be points such that O,A,B,C are not coplanar. The (3-dimensional) lattice Λ with basis $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ consists of all points P such that

$$\overrightarrow{OP} = u\overrightarrow{OA} + v\overrightarrow{OB} + w\overrightarrow{OC}$$

for some *integral u, v, w*. The parallelepiped Π determined by O and the basis vectors is the solid body made up of all points Q such that

$$\overrightarrow{OQ} = x\overrightarrow{OA} + y\overrightarrow{OB} + z\overrightarrow{OC}$$

for real numbers x, y, z in the interval $[0, 1]$.

A parallelepiped is a *cell* of Λ if its 8 vertices are all lattice points (i.e. points of Λ) and it has the same volume as Π. In particular, Π itself is a cell, and every cell with O as a vertex is obtained similarly from some basis $\overrightarrow{OD}$, $\overrightarrow{OE}$, $\overrightarrow{OF}$.

A lattice is *unimodular* if its cells have volume 1. Since every lattice is just an expanded or contracted version of a unimodular one, it is often appropriate to confine attention to unimodular lattices. Two lattices are said to belong to the same *congruence class* if they are congruent in the usual sense of Euclidean geometry and, if so, their cells have the same volume.

We now look more closely at the geometry of an individual lattice Λ. Expand each lattice point to a sphere of fixed radius r, assuming for the moment that no two spheres overlap. Then the proportion of space covered by the spheres is just the volume of a sphere divided by the volume of a cell. Even when the spheres overlap, this number still makes sense as a 'covering ratio'.

Two cases are of particular interest. They are related to two lattice constants $M = M(\Lambda)$ and $m = m(\Lambda)$. To simplify the present discussion, we assume that Λ is unimodular, so that the covering ratio above becomes simply $4\pi r^3/3$.

The more obvious constant, M, is the minimum squared distance between distinct lattice points. When $r = M^{1/2}/2$, certain of the spheres will touch but no two will actually overlap. In other words, this value of r provides the *closest packing* of equal spheres with centres at the lattice points. The corresponding *packing density* is $4\pi(M^{1/2}/2)^3/3 = \pi M^{3/2}/6$.

The mathematical problem that arises here is to determine the congruence class (or classes) for which the packing density is greatest. Gauss solved this problem in 1831, confirming the conjecture that the maximum packing density is $\pi\sqrt{2}/6 = 0.74\cdots$.

In defining the other constant, m, we borrow an image from Conway-Sloane: the points of space are where children live, schools are at the lattice points and a child attends the nearest school; $m^{1/2}$ is then the greatest distance any child has to go to school. A little thought shows that $r = m^{1/2}$ is the smallest radius for which the spheres cover the whole of space. In other words, this value of r yields the *thinnest covering* of space by equal spheres with centres at the lattice points. The *covering density* is $4\pi m^{3/2}/3$. The corresponding mathematical problem turns out to be harder than the packing problem. Only in 1954 was it shown by Bambah that the minimum covering density is $5\sqrt{5}\,\pi/34 = 1.46\cdots$.

The above ideas can be translated into the language of coordinate geometry. This provides the link to positive quadratic forms and is an essential step in generalizing the theory to higher dimensions.

Let Λ be a lattice with basis $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ as described earlier. Real coordinates $\mathbf{x} = (x, y, z)$ are then assigned to each point P of space via the vector equation

$$\overrightarrow{OP} = x\overrightarrow{OA} + y\overrightarrow{OB} + z\overrightarrow{OC} ,$$

the points of Λ itself being those with integral coordinates. If the basis vectors are mutually orthogonal vectors of length 1, we have an ordinary rectangular Cartesian coordinate system and the distance OP is given by the familiar formula $OP^2 = x^2 + y^2 + z^2$. In general, OP^2 is a more complicated quadratic form $f(\mathbf{x})$ in x, y, z. By its meaning, f is *positive*. It is a standard result of linear algebra that *every* positive quadratic form arises in this way from a suitable lattice Λ.

We say that the form f above is a *distance function* of Λ and that Λ is a lattice *associated with f.* It can be shown that the determinant of f is the square of the volume of a cell of Λ. Accordingly, f is called *unimodular* if its determinant is 1.

Consider another basis $\overrightarrow{OD}$, $\overrightarrow{OE}$, $\overrightarrow{OF}$ of Λ and let the new coordinates of P and corresponding distance formula be $\mathbf{X} = (X, Y, Z)$ and $OP^2 = F(\mathbf{X})$. The general relation between $\mathbf{X}$ and $\mathbf{x}$ is that each of X, Y, Z is an *integral* linear combination of x, y, z and vice versa. The forms f and F are accordingly equivalent in the sense used in earlier sections. The forms equivalent to f make up its *equivalence class*.

These considerations suffice to translate problems about lattices into problems about positive quadratic forms. It can be seen that, if f is a distance function of a unimodular lattice Λ with lattice constants $M(\Lambda)$ and $m(\Lambda)$, then $M(\Lambda) = M(f)$ and $m(\Lambda) = m(f)$, where $M(f)$ and $m(f)$ are the homogeneous and inhomogeneous minima of f as defined in earlier sections. Therefore the packing problem for lattices is equivalent to determining those equivalence classes of unimodular positive quadratic forms for which the homogeneous minimum is greatest. The covering problem translates similarly.

We conclude with a brief indication of how the 3-dimensional theory is generalized to higher dimensions. By definition, the points P of n-dimensional Euclidean space, R^n, are the n-tuples $\mathbf{x} = (x_1, \ldots, x_n)$ of real numbers and distances from the origin O are given by $OP^2 = x_1^2 + \ldots + x_n^2$. Now let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be an algebraic basis of R^n, so that each $\mathbf{x} = X_1\mathbf{a}_1 + \ldots + X_n\mathbf{a}_n$ with unique coefficients $X_1,\ldots X_n$. Then $\mathbf{X} = (X_1, \ldots, X_n)$ is the coordinate vector of P with respect to this basis and OP^2 is a positive quadratic form $F(\mathbf{X})$ in the new coordinates. Those points P whose coordinates $X_1,\ldots,X_n$ are integers form the *lattice* with the above basis.

All that has been said about the 3-dimensional case carries over in a reasonably straightforward way to the n-dimensional case. In particular, the counterparts for positive quadratic forms of the covering and packing problems for lattices are classical problems of number theory.

Barnes's papers on positive quadratic forms and lattices will now be discussed under the appropriate subject headings.

Lattice packings of spheres

All but one of Barnes's nine papers in this area were written in the period 1955–58 during his time at the University of Sydney. The ninth, a joint paper with Sloane, appeared in 1983, the year of Barnes's retirement. Although related to the last of the earlier papers, its background is in the theory of error-correcting codes, a subject that came to maturity only in the intervening decades. Three of the papers are discussed at some length below.

The perfect forms in 6 variables. Let f be a positive quadratic form in n variables and Λ an associated n-dimensional lattice. Those f for which the packing density of Λ is either absolutely or locally maximal are of particular interest. Thus, f is called *absolutely extreme* if Λ provides the densest possible lattice packing of spheres in R^n, *extreme* if no lattice obtained by slightly deforming Λ provides a denser lattice packing. The *perfect* forms referred to above are a somewhat wider class than the extreme forms, and their precise definition need not be given here.

We briefly explain what the above definitions mean in purely algebraic terms. With f as above, write $\gamma(f) = MD^{-1/n}$, where $M = M(f)$ and $D = D(f)$ are the homogeneous minimum and determinant of f. Then $\gamma(f)$ is a function of the coefficients of f: it achieves its absolute maximum at the absolutely extreme forms and its local maxima at the extreme forms.

The absolute maximum just referred to is *Hermite's constant* γ_n. Its precise value is known only for $n \leq 8$. By the 1950s there had been extensive research by many writers on γ_n and on perfect, extreme and absolutely extreme forms. The determination of the perfect forms in three variables goes back to Gauss in 1831 and in 4, 5 variables to Korkine and Zolotareff[24] in 1877. Hofreiter[25] in 1933 claimed to have determined the extreme forms in 6 variables but his list turned out to be both erroneous and incomplete. Barnes [(25), (26)] closed a chapter in this history by determining the *perfect* forms in 6 variables.

Twenty years later, Barnes's results were confirmed, by a different method and with the aid of a computer, by Stacey[26], who herself[26,27] made considerable progress in classifying the perfect forms in 7 variables. The latter were completely classified only in 1991. The problem for 8 variables remains unsolved. The numbers of essentially different perfect forms in 6 and 7 variables are 7 and 31 respectively. For 8 variables, the number is known to be *at least* 10,916 (Martinet,[28] p.218).

The method that Barnes uses, due to Voronoi,[29] is essentially geometrical. Although it applies to n-variable forms in general, we confine attention here to the case $n = 6$. A positive quadratic form in 6 variables has 21 coefficients and is therefore representable by a point in 21-dimensional space R^{21}. Each unimodular perfect form f determines (in a certain way that will not be explained here) a 21-dimensional cone $V(f)$ in R^{21}. Each such *Voronoi cone* is bounded by a finite number of 'flat' 20-dimensional faces, called its 'facets'. A given facet of $V(f)$ is also a facet of exactly one other Voronoi cone $V(f')$ and the finitely many unimodular perfect forms f' obtained in this way are the *neighbours* of f.

We are now in a position to describe Voronoi's method of determining the perfect forms in 6 variables. A list of forms $f_1,\ldots,f_k$ is constructed step by step. The first entry f_1 may be *any* unimodular perfect form (examples are known). The neighbours of f_1 are then determined one by one and, if not equivalent to a form already there, entered onto the list. The procedure is then repeated with the neighbours of f_2, and so on. After a finite number of steps, no further new forms are produced and the list is complete.

Voronoi's method, although straightforward in principle, presents considerable difficulties in practice. Each perfect form has at least 21 neighbours and in determining each of them it is necessary to solve certain systems of linear inequalities in 21 variables. Barnes modified Voronoi's procedure in a simple but ingenious way that takes advantage of the symmetry of the associated lattice. Even so, determining the 36 neighbours of the absolutely extreme form proved very difficult and Barnes's treatment of this case is a tour de force of combinatorial algebra.

Dense lattice packings in large dimensions. In contrast to (25), the Barnes-Wall paper (30) is essentially concerned with forms in a large number of variables. In what follows, f_n stands for a positive quadratic form in n variables and $\gamma(f_n)$, as earlier, for its 'scaled homogeneous minimum' $MD^{-1/n}$, where $M = M(f_n)$ and $D = D(f_n)$.

We recall that the greatest possible value of $\gamma(f_n)$, assumed when f_n is absolutely extreme, is Hermite's constant γ_n. Although its precise value is known only for small n, there are quite good estimates of its ultimate size: it lies between $n/2\pi e$ and $n/\pi e$ for sufficiently large n. In this sense, 'γ_n has order n as $n \to \infty$'.

Let n_1, n_2, … be a strictly increasing sequence of positive integers. Consider now a corresponding infinite sequence of forms obtained by taking a positive form f_n in n variables for $n = n_1, n_2, \cdots$. The above considerations show that the corresponding numerical values $\gamma(f_n)$ *could* be of order n as $n \to \infty$. However, in all such sequences of forms *explicitly* constructed up to 1959, the values $\gamma(f_n)$ turned out to be *bounded*, that is, all were less than some fixed value C. The achievement of (30) was to construct for the first time a sequence of forms for which the numbers $\gamma(f_n)$ are *unbounded* — in fact, of order $n^{1/2}$ for large n.

The values n_i chosen in (30) are the powers 2^i. For each such value, a number of forms are constructed, many of them extreme. For $n = 2^i$, we now understand f_n to mean the 'best' of the forms constructed in 2^i variables. For these forms, the value of $\gamma(f_n)$ is in fact $(n/2)^{1/2}$.

The forms f_n with $n = 2^i$ are of interest for small, as well as large, i. Thus, f_4 and f_8 are known to be absolutely extreme, while f_{16} is conjectured to be so. The forms f_{32} and f_{64} also provide dense lattice packings, although these have now been surpassed.

The papers (27), (28) should be mentioned here. Various infinite sequences of extreme forms had long been known. Barnes used uniform methods to construct further such sequences. Of particular interest is his method of constructing forms in $n + 1$ variables from known forms in n variables. A comprehensive generalization of Barnes's work was made by former research student Scott.[30, 31]

Lattice packings constructed from codes. Over 20 years elapsed between the paper (30) discussed above and Barnes's final paper on sphere packing (45). Although this was not realized at the time, the former may be viewed as an application of the Reed-Muller codes to packing theory. The latter, written jointly with Sloane, is concerned with the *systematic* application of coding theory to the construction of lattice packings of spheres.

Coding theory originated in the late 1940s and by the 1960s was widely recognised as an independent subject. Leech[32, 33] in 1964 and 1967 made explicit use of codes to construct dense packings in dimensions 2^n and 24, respectively. Generalizing Leech's work, Leech and Sloane in 1971 gave general methods for manufacturing packings of equal spheres out of codes. (This paper is reprinted as Chapter 5 of Conway-Sloane.[23]) Many of the resulting packings were non-lattice (where the centres of the spheres do not form a lattice) and they included ones denser than any previously known.

The paper (45) is similar in purpose and general structure to the Leech-Sloane paper, except that it is concerned entirely with *lattice* packings. There are spectacular applications. For example, starting out from the famous Leech lattice in R^{24}, lattices are constructed in R^n for every multiple n of 24 up to 98,328. Apart from a few lower-dimensional exceptions, the corresponding lattice packings are the densest known in their dimensions.

Lattice covering by spheres

Of Barnes's three papers in this area, the first, which appeared in 1956, was his first substantial work on positive forms. The other two, written some time later, were joint papers with former research students Dickson (1967) and Trenerry (1972).

The absolutely extreme and extreme lattices and forms of packing theory have natural counterparts in covering theory. Barnes calls them by the same names with the qualification 'in the sense of covering theory'. To avoid confusion here, we call them *optimal* and *locally optimal*, respectively.

The locally optimal ternary forms. Barnes's paper (23) is concerned with lattices in R^3 or equivalently with ternary positive forms. At the time of writing of the paper, the ternary *extreme* forms had long been known but a ternary *optimal* form had been discovered for the first time by Bambah[34] in 1954. Barnes sharpens Bambah's result by proving that every *locally optimal* ternary form is equivalent to a multiple of Bambah's form.

Barnes uses a second geometrical method due to Voronoi.[35] Consider a 3-dimensional lattice Λ and a point P of Λ. Those points of space for which P is the closest lattice point form a certain region Ω. By its meaning, the covering radius $r = r(\Lambda)$ is the greatest distance of any point of Ω from P. The region Ω is a *polytope* and consequently the points of Ω farthest from P are among its finitely many vertices.

In the light of these considerations, the paper proceeds as follows. The geometry is used first to express r, and thence the covering density $\delta = \delta(\Lambda)$, as functions of the coefficients of a suitable distance function f for Λ. Algebra is then used to determine those values of the coefficients at which the function δ has a local minimum. As usual in Barnes's work, this programme is carried out with exceptional skill and elegance.

The *locally optimal* forms in 4 variables were subsequently determined by Dickson[36] in 1967 and the *optimal* forms in 5 variables by Ryškov and Baranovskiǐ[37] in 1975.

Covering theory is more difficult than packing theory and less is known about it. The papers of Barnes and Dickson (33) and Barnes and Trenerry (35) make significant contributions to *general* covering theory.

A basic theorem of Voronoi[29] clarifies the relation between perfect and extreme forms. An analogous theorem in covering theory is proved in (33). Essential use of (33) is made by Dickson in his determination of the locally optimal forms in 4 variables.

Various infinite sequences of extreme forms arise naturally and have been known at least since Korkine and Zolotareff's work in the 1870s. The situation for locally optimal forms is, however, quite different. An infinite sequence of such forms was first constructed by Bleicher[38] in 1962. Barnes and Trenerry (35) construct a second such sequence of surprisingly complicated form.

Lattice quantizers

The one paper on this subject was the joint paper (46) with Sloane, published in 1983.

The simple process of rounding off numbers to the nearest integer can be formulated in lattice terms. The integers 0, ± 1, ± 2, … form the lattice Z^1 on the real line R^1. Rounding off a number r in R^1 to the nearest integer means replacing it by the nearest lattice point. It can be shown that the average squared error incurred in this process is 1/12.

Now let Λ be a lattice in R^n. The associated process of *quantization* replaces each point of R^n by the nearest point of Λ. Let $\tilde{G}(\Lambda)$ denote the resulting average squared distance error. Making technical adjustments for dimension and scale, one arrives at an 'absolute' measure $G(\Lambda)$. The smaller the value of $G(\Lambda)$, the *more efficient* Λ is said to be as a quantizer.

The value of $G(\Lambda)$ for the plane lattice Z^2 (and indeed for Z^n in general) is $^1/_{12} = 0.08333\cdots$. However, the most efficient plane lattice is actually the hexagonal lattice, for which $G(\Lambda) = 5/(36\sqrt{3}) = 0.08018\cdots$.

In general terms, the larger the dimension the more efficient lattices can be. Let G_n denote the smallest possible value of $G(\Lambda)$ for lattices Λ in R^n. Zador[39] in 1963 proved the remarkable result that G_n tends to the value $1/2\pi e = 0.05855\cdots$ as n tends to infinity.

Despite Zador's result, actually constructing highly efficient lattices in R^n remains a difficult problem. For example, the *q-optimal* lattices in R^n for which $G(\Lambda)$ attains the minimum possible value G_n have been determined only for $n = 1, 2, 3$. The case $n = 3$ is due to Barnes and Sloane (46). Their work yields the value $G_3 = 19/(192.2^{1/3}) = 0.07854\cdots$, which is roughly 2% smaller than G_2.

In general outline, the proof by Barnes and Sloane follows Barnes's proof in (23) rather closely. Indeed, in the quantization process associated with a lattice Λ in R^3, the polytope Ω about a lattice point P consists *precisely* of the points rounded off to P. Moreover, the average, $\tilde{G}(\Lambda)$, taken over R^3 is the same as the average taken over Ω. This is calculated by an ingenious argument, which involves dissecting Ω into 60 tetrahedra. Both hand and computer calculations are involved.

Reduction of positive forms

The five papers briefly reviewed here were published in the period 1975–82.

We are concerned with positive quadratic forms in n variables. Insight into the totality of such forms may be gained by picking out one or more representative forms from each equivalence class. 'Hermite reduction' and 'Minkowski reduction' are two methods for doing this. *Reduction theory*, which deals with such methods in general, is one of the most important (and oldest) branches of the arithmetical theory of positive forms.

A *reduced set* of forms is one such that every form is equivalent to at least one, and at most finitely many, of its members. The region in R^N $[N = n(n + 1)/2]$ formed by the coefficient vectors of the members of such a set is called a *reduction domain*. A *fundamental domain* is a reduction domain with the stronger property that no two of its interior points represent equivalent forms.

Minkowski defined certain reduction and fundamental domains Φ and Φ^+, both of which are convex polyhedral cones. Barnes and Cohn (37) explicitly determined their edges (i.e. 1-dimensional faces) when $n \le 4$: Φ has 323 edges and Φ^+ 109 when $n = 4$. Heavy calculations (partly by computer) are involved.

It is natural to ask whether alternative methods of reduction might lead to simpler domains. Again for $n = 4$, Barnes and Cohn (38) construct a fundamental domain with just 12 edges. It is constructed in an ingenious way from Voronoi cones.

It was proved by Minkowski himself that there is a constant λ_n such that the inequality $a_{11}a_{22} \ldots a_{nn} \le \lambda_n$ holds for every unimodular Minkowski-reduced form

$$f(x_1, \ldots, x_n) = \Sigma a_{ij} x_i x_j.$$

Several mathematicians observed that, in the cases $n = 2, 3$, this can be sharpened by replacing the left hand side of the inequality by a more complicated function of a_{11}, $\ldots$ a_{nn}. In (42), Barnes extends these results to the case $n = 4$, giving a clear explanation of how such inequalities come about. The results of Barnes and Cohn (37) are required in the calculations.

What is involved in refining Minkowski's inequality for general n was further clarified by Barnes in (43). Finally, Barnes and Trenerry (44) showed that *no* such refinement exists when $n = 5$. There is, however, a refinement that holds when a_{55} is large enough compared with the other diagonal elements. Heavy algebra is required in the analysis and several natural questions are left open.

Concluding Remarks

Eric Barnes was internationally recognised as a leading contributor to the growth of knowledge in the geometry of numbers. The continued relevance of his outstanding work can be seen from recent publications in the area, as well as from the two main general reference works[4, 23] and the recent more specialized book of Martinet.[28] Through his research, teaching, scholarly work and professional service, he made a major contribution to the development of research and education in mathematics in Australia.

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