CSIRO PUBLISHING

Historical Records of Australian Science, 2019, **30**, 49–57
https://doi.org/10.1071/HR18012

George Szekeres 1911–2005

M. G. Cowling[A,B], *D. C. Hunt*[A] *and J. D. Steele*[A]

[A]School of Mathematics and Statistics, University of New South Wales, UNSW Sydney, NSW 2052, Australia.
[B]Corresponding author. Email: m.cowling@unsw.edu.au

George Szekeres was a distinguished Hungarian-Australian mathematician, who worked in many different areas of mathematics, and with many collaborators. He was born in Budapest in 1911. His youth between the two World Wars was spent in Hungary, a country that, as a result of historical events, went through a golden age and produced a great number of exceptional intellects; his early mathematical explorations were in the company of several of these. However, for family reasons, he trained as a chemist rather than a mathematician. From 1938 to 1948, he lived in Shanghai, China, another remarkable city, where he experienced the horrors of persecution and war but nevertheless managed to prove some notable mathematical results. In 1948, he moved to Australia, as a lecturer, then senior lecturer, and finally reader, at the University of Adelaide, and then in 1964 he took up the Foundation Chair of Pure Mathematics at the University of New South Wales; in Australia he was able to bring his mathematical talents to fruition. After many years in Sydney, he returned to Adelaide, where he died in 2005. We discuss his early life in Hungary, his sojourn in Shanghai, and his mature period in Australia. We also discuss some aspects of his mathematical work, which is extraordinarily broad.

Published online 6 December 2018

Biography

George Szekeres (Szekeres Győrgy in Hungarian, Fig. 1) was born in Budapest, on 29 May 1911, the second of the three sons of Ármin and Margit Szekeres (née Zipser). (We will use the English form of George's name throughout, although he himself used both forms. Similarly, for other Hungarians, we will often use the English version of their names, but mention the Hungarian form when they first appear.) The Szekeres family, which was Jewish, owned a business tanning leather, which had made boots for the troops in the First World War, and was very well off as a result. We begin by sketching a little of the history of Hungary, to illuminate the world in which he grew up.

Budapest was a thriving city. After the Hungarians rebelled against the Austrian emperor in 1848, the Habsburg monarchy responded by changing the name from Austrian Empire to Austro-Hungarian Empire, naming Budapest as a co-capital of the Empire, and strengthening the Hungarian economy, so much so that by World War 1 the population of Budapest had quadrupled, and it had the largest stock market and the first underground railway system in continental Europe. Maurice von Kármán led a reform of Hungarian schools that aimed to incorporate the best elements of the French and German systems. Jews were emancipated in 1867, and many of the arrivals in the capital were Jewish; many names were changed to become Hungarian (the Szekeres family were called Schleininger before Ármin changed the name). All this led to Budapest becoming an intellectual hotspot; by way of illustration, the polymath John von Neumann, the physicists Leo Szilárd and Edward Teller, the film director Alexander Korda, the photographer Robert Capa, the musicians George Széll and George Solti, and the writer Arthur Koestler were all educated there in the late 1800s or early 1900s. Arguably, the spirit of the age is captured in a quote from Count Károly: 'We keep the Gypsies to play music for us, since we are too lazy to do it for ourselves, and the Jews do the work for us'.[1] However, the short-lived 1919 Bela Kun soviet republic and the rise of Nazism in the 1920s led to increasing anti-Semitism, and many of these exceptional people emigrated.[2]

Figure 1. George Szekeres. Photograph: David Harvey.

[1] Hanák (1999) p. 18.
[2] Much more is available about the historical period and the people who graced it. See, for instance, Hanák (1999). Hargittai (2006). Marton (2006).

George Szekeres was a retiring youth who was fascinated by the problems in mathematics and physics that he found in the magazine *Középiskolai Matematikai és Fizikai Lapok* for high school students. He was particularly inspired by the teaching of his physics teacher Charles (Károly) Novobátzky, who stimulated George's lifelong interest in relativity. Novobátzky evidently knew his subject very well, as he later became professor of theoretical physics at the University of Budapest. When George finished high school, the Szekeres family needed a chemist in the family business, and so George followed his parents' wishes and studied chemical engineering at the Műegyetem (Budapest University of Technology and Economics). He took out his degree in 1933; this entailed only one mathematics course, on calculus.

George's high school friend Paul (Pál) Turán, who studied mathematics at university, introduced him to a group of enthusiasts who met regularly in the City Park in the early 1930s to discuss mathematics and solve problems. George quickly showed his ability. One of his contributions was to explain Hermann Weyl's *Raum, Zeit, Materie* to the others. Apart from Turán, the group included Paul (Pál) Erdős, who would go on to become the most prolific mathematician ever, Tibor Gallai (born Grünwald), Geza Grünwald, and Esther (Eszter) Klein and Marta (Márta) Wachsberger, who were friends from high school. Since there were limits on the number of university places for Jewish students, Esther was studying physics while Marta studied mathematics. The group worked through *Aufgaben und Lehrsätze aus der Analysis* by Pólya and Szegő, and also solved problems that they posed themselves. Much has been written about this unusual group of mathematics aficionados.[3]

One of the problems that the group considered was posed and solved by Esther: given five points in a plane, no three of which are collinear, prove that there are four of them that determine a convex quadrilateral. George and Paul Erdős generalised this problem,[4] which Erdős dubbed the Happy Ending Problem, because at this time it became clear that George and Esther were perfectly matched.

They were married on 13 June 1937, but could not afford to live together, as Esther was in Budapest while George was working in Simontornya, one hundred kilometres away; rather they spent weekends together. In due course, they would have two children, Peter, born in 1940, and Judith (Judy), born in 1954. Following in his father's footsteps, Peter later became a mathematical physicist, with a focus on relativity, while Judy became a musician and university administrator.

George did not enjoy coming from a wealthy family, and was curiously relieved when, during his university studies, the family firm failed; indeed, he became more extroverted. He worked as an analytical chemist in Hungary from 1933 to 1939. The rise of Hitler and native Hungarian anti-Semitism made Europe an increasingly dangerous place for George and Esther, and so they emigrated when George's brother Imre, who was already in Shanghai, helped find him a job there. Again, some background about this remarkable city may be useful.

Shanghai was an open city, and no visa was required to go there. It had long been a major port, but the Chinese rejected international trade for a long period. Beginning with the First Opium War in 1840 and continuing with various treaties in the later part of the 1800s, it became one of the main commercial cities in China, and home to various European and American 'concessions', where a substantial number of émigrés lived, initially merchants and their families, but over time many of these acquired other interests. By 1932, it was the fifth city in the world by population. Famous 'Shanghailanders' include W. Michael Blumenthal (USA Treasury Secretary 1977–9), the ballerina Dame Margot Fonteyn, and the writer J. G. Ballard.

The Japanese laid siege to Shanghai in 1932, took over most of the city in 1937, and occupied the foreign concessions in 1941. In the 1930s, Shanghai was already home to tens of thousands of Jews, both wealthy Sephardi merchants from the Middle East who had moved there in the 1800s and Ashkenazi refugees from Russia who had arrived in the early 1900s; another fifteen thousand refugees from Europe arrived in the late 1930s. The Jewish population of Shanghai was about fifty thousand.[5]

George and Esther left Hungary in late 1938 and arrived in China in 1939—out of the frying pan and into the fire! Japan had already invaded Shanghai, and life was very difficult, particularly after 1943, when the Japanese, in response to German pressure, started actively persecuting Jews. George used to reminisce about swapping his bicycle for a sack of rice, and Peter Szekeres remembers George picking him up and running to avoid bombing. Imre died from an infection, and George feared that the whole family would perish; he was greatly relieved when the war ended very quickly after the deployment of the atomic bomb. After the war, George worked as a clerk at an American airbase. Despite these difficulties, he managed to do mathematics and meet mathematicians in China.

While the Szekeres family was still in Hungary, Esther's friend Marta Wachsberger married George Sved (Svéd), another brilliant mathematician who had become an engineer, who took a job at the University of Adelaide in the Department of Civil Engineering. George Sved brought George Szekeres's name to the attention of H. W. Sanders, Elder Professor of Mathematics at the University, who played an important role in offering George a job in Adelaide. No doubt the task of appointing George was made easier by the impressive referees from different areas of mathematics who supported his application, including the geometer S. S. Chern and the number theorist L. J. Mordell. Graeme Cohen describes Szekeres' arrival in Adelaide, and reports Ren Potts' recollections of the administrative difficulties associated with George's lack of formal mathematical qualifications.[6]

The Szekeres family (now with Anglicised names) moved to Adelaide in June 1948; there they shared a flat with the Sved family for three years. While Adelaide was very different from Budapest or Shanghai, the family soon fell in love with Australia, which George much later described as very civilized, and with the Australian bush. Alhough George started as lecturer at the University of Adelaide, he rose rapidly through the ranks to become senior lecturer (1950) and

[3] See, for instance, the biographies of Erdős: Hoffman (1998). Schechter (2000).
[4] Erdős and Szekeres (1935).
[5] For much more about Shanghai and its Jewish community, see: Kranzler (1976). Pan (1995).
[6] Cohen (2006) pp. 175–176.

reader (1957), due to his outstanding research. He was elected to Fellowship of the Australian Academy of Science in 1963; Eugene Seneta recalls Eric Barnes interrupting one of George's lectures to bring the news of his election. At this time, Esther was tutoring at the University of Adelaide.

In 1963, George was recruited by John Blatt to UNSW as Foundation Professor of Pure Mathematics, and arrived in May 1964. Before accepting the post, he requested and received assurances that he would not be expected to carry out any administration; despite this, he was acting head of school for a brief period shortly after his arrival. Esther initially found work at the University of Sydney, and then established a long-term relationship with Macquarie University, where she was later awarded an honorary DSc. The family bought the house of the recently deceased physicist Gilbert Bogle (of the Bogle–Chandler mystery),[7] in Turramurra, which was on a large block adjacent to bushland. George wrote to a friend that they had 'found paradise'.

George was a member of the Australian Mathematical Society from its foundation on 15 August 1956. He served as president from 1972 to 1974 and as vice-president for the years 1971–2 and 1974–5. In recognition of George's achievements, an issue of the *Journal of the Australian Mathematical Society*, edited by John Giles and Jenny Seberry, was dedicated to him when he retired.[8] Much more significantly, in 2001, the Society created a new award, its most prestigious, and named it the Szekeres Medal. Other recognition of George's successful career includes the awarding of the Lyle Medal by the Australian Academy of Science in 1968, an Honorary DSc from the University of New South Wales in 1977, and the Order of Australia Medal in 2002 for service to Mathematics. George himself was perhaps proudest of being admitted as one of the very few foreign members of the Hungarian Academy of Science in 1986. It is arguable that his part in the creation of one of the most successful schools of mathematical sciences in Australia is one of his greatest achievements.

While at high school, George had taken part in mathematics and physics problem solving and competitions, and he was very active in promoting similar activities for mathematics students in Australia. Together with Jim Williams of the University of Sydney, he was instrumental in setting up a training program in Australia for the International Mathematical Olympiads (at which Australia has performed very well); George was the Deputy Team Leader for the first two competitions at which the Australian team competed (in 1981 and 1982). Together with Esther and Terry Gagen (also of the University of Sydney), he organised weekly problem-solving sessions at Mercy College, Chatswood, which have run continuously from 1984; these sessions have also been offered in other venues. He founded the magazine *Parabola* for high school students, which continues to this day. George and Esther were sources of many problems for Chatswood and *Parabola*, as well as for the University of New South Wales School Mathematics Competition and the Sydney University Mathematics Society Undergraduate Mathematics Competition.

George retired in 1976, at 65 years of age, but he continued to attend UNSW almost every day for the next twenty-five years and published some twenty papers in his retirement. He continued to mentor young mathematicians, through the problem-based activities mentioned above; these young mathematicians include the first Australian Fields Medal winner, Terence Tao. At the same time, he continued to be active in music and bush-walking. He climbed Pigeonhouse Mountain near Nowra in his seventies and walked, section by section, the whole Great North Walk between Sydney and Newcastle in his eighties, with first Judy and later John Giles. George played violin and viola, and he was a member of the Ku-ring-gai Philharmonic Orchestra, from its foundation in 1971 up to 2000, when he decided that he was no longer dextrous enough to keep playing with the orchestra. There was a violoncello in the UNSW School of Mathematics and Statistics that was used by various visitors to play chamber music with George and his friends.

In 2004, George's driving license was not renewed, and living in Turramurra several kilometres from the nearest shops became impractical, so George and Esther decided to move back to Adelaide, close to both Peter and Judy. They bought a house there, but perhaps under the stress of the move, Esther had a stroke, and moved straight to the Wynwood Nursing Home; George joined her there after less than a year. On 28 August 2005, at 6.15 a.m., George died, and Esther died before 7.00 a.m. on the same day. Their remarkable lives and close deaths led to obituaries of both in many newspapers.[9] More mathematical obituaries appeared in the *Australian Mathematical Society Gazette*[10] and *Mathematikai Lapok* (the journal of the János Bolyai Mathematical Society; this memoir was later reproduced by the Hungarian Academy of Science).[11]

Scientific achievements

George Szekeres' unusual background had some interesting consequences that he described as follows:

> Lack of formal mathematical education had some obvious drawbacks, but also some beneficial effects. I was forced to pick up practically all mathematical knowledge on my own initiative, without directed guidance from my elders. Apart from books, my main source of enlightenment came from the stimulating environment of several exceptionally gifted young students …. In exchanges with them, I learned very soon what makes mathematics tick, probably far better that I could have from formal education. Not being directed in any particular channel, which is almost unavoidable when one works for a Ph.D., I developed a taste for a much wider spectrum of mathematics than most mathematicians do when they go through the conventional avenues of education.[12]

And others shared his views. We quote from the citation for his Lyle Medal:

> Professor G. Szekeres has published, in the period 1964–1968, thirteen research articles in a wide range of mathematical disciplines. Of particular significance are his papers giving the solution of a long outstanding problem on the fractional iterates of an entire

[7] Butt (2012).

[8] *Journal of the Australian Mathematical Society. Series A*, **21** (1976).

[9] Cowling (2005*b*). 'MATP' (2005). Morgan (2005*a*). Morgan (2005*b*).

[10] Cowling (2005*a*).

[11] T.-Sós and Laczkovich (2004/2005).

[12] *Curriculum vitae*, George Szekeres files, University of New South Wales Archives.

> function; a new practical method for computer evaluation of high dimensional integrals that eliminates some of the difficulties of the standard Monte Carlo methods; the solution of a problem in the theory of directed graphs which has pointed the way to new work in this field; and a new system of axioms for relativistic kinematics. These display a mastery of the concepts and techniques of analysis, algebra, numerical analysis, combinatorics and mathematical physics which is seldom found in one individual.

Cohen mentions another indication of George's mathematical breadth: in 1968, a review of pure mathematics in Australia found that George was 'expert' in three subfields, more than any other pure mathematician in the country;[13] however, that conclusion did not take into account his contributions in numerical analysis or physics!

We now describe the main areas of his work and some of his most important contributions in more detail, although given the breadth of his work, it is not possible to include everything, and we have attempted to choose representative examples of the main research areas that he examined. It is often said that mathematics is about patterns, and this is certainly one of the major themes in George's work. These patterns are sometimes geometric, sometimes algebraic, sometimes analytic, and sometimes physical. But George was always interested in the questions of what kinds of patterns can appear, and how can we classify them.

Graph theory

A substantial part of George's mathematical work is in the area known as graph theory. A graph is a mathematical abstraction of a network: it is made up of points, usually known as vertices, and edges that join some of the pairs of vertices. Graphs are used in many areas of application, including chemistry, where the atoms of a molecule are represented by the vertices of the graph, and the bonds between them by edges, computer network design, where the vertices represent computers and the edges physical links between them, and epidemiology, where the vertices represent people and the links a physical contact capable of passing on a disease.

Vertices that are linked by an edge are said to be adjacent, while those not linked by an edge are said to be independent; a clique is a collection of vertices in which every pair is linked by an edge. In 1928, the English polymath Frank Ramsey (1903–30) proved an important result,[14] now called Ramsey's Theorem, which may be stated as follows: 'there is a smallest number, $R(m,n)$ say, with the property that every graph with at least $R(m,n)$ vertices either contains a clique with at least m vertices or a collection of at least n independent vertices'. For instance, $R(3,3)=6$, so in any group of 6 or more persons, there are at least 3 who are mutual friends or 3 who are mutual strangers. The Ramsey numbers $R(m,n)$ are rather mysterious: it is possible to give upper and lower bounds for them, but except in a few cases, they are not known. The significance of Ramsey theory is that it can be rephrased in many ways, all of the form 'given a large number of objects, either a certain number of these have a lot of structure or a certain number have very little structure'.

George's best-known paper (or at least his most cited paper) is joint with Paul Erdős, about the 'Happy Ending Problem'.[15] As already mentioned, this concerns an extension of a geometric problem posed and solved by Esther, which may be stated as follows: 'Suppose that you are given five points in the plane. Show that you can always pick four of these that form the vertices of a convex quadrilateral.'

The extension of the problem comes when one considers more than five points and tries to find some points that form the vertices of a convex pentagon or hexagon or n-gon (which is a polygon with n sides). Erdős and Szekeres showed that for any n, if you are given N points in the plane, where N is big enough, then you can always pick n of these that form the vertices of a convex n-gon. They conjectured that $N=2^{n-2}+1$ is enough, but could not prove this. However, they did manage to construct a set of 2^{n-2} points, which contains no convex n-gon,[16] adding to the plausibility of their conjecture.

Quite recently, Peters and Szekeres gave a computer proof that this conjecture is correct when $n=6$ and $N=17$;[17] this was quite a tour de force, as the number of configurations to examine is huge. While this problem appears to be purely geometric, it may also be formulated in graph theoretic terms.

Erdős and Szekeres gave two solutions to the Happy Ending Problem; the first of these used Ramsey's theorem, and the second was more geometric. Along the way, they rediscovered and extended Ramsey's work and obtained an upper bound for Ramsey numbers that is still used today:

$$R(m,n) \le \binom{m+n-2}{n-1}$$

(the right-hand expression is a binomial coefficient). It is perhaps worth observing that the Erdős–Szekeres bound states only that $R(5,5) \le 70$; it is now known that $R(5,5)$ lies between 43 and 48, so it seems that there is still a significant gap between the best theoretical upper bounds for general Ramsey numbers and their actual values.

In a graph, a path is a sequence of moves, each of which goes from one vertex to an adjacent one; these can represent the transmission of data in a network or disease in a population. A graph is connected if any two distinct vertices are joined by a path. A tree is a connected graph without circuits; that is, there is no path of at least three edges whose first and last vertices coincide but all the other vertices are distinct. A rooted tree is one in which a particular vertex is chosen and called the root; the height of a rooted tree is the length of the longest path without back-tracking from the root to another vertex. These are used to model data structures in computer science, and their height is an indication of how long it takes to move around the structure.

The nineteenth-century mathematicians Carl Wilhelm Borchardt and Arthur Cayley showed that there are exactly n^{n-2} distinct

[13] Cohen (2006) p. 267.
[14] Ramsey (1929).
[15] Erdős and Szekeres (1935).
[16] Erdős and Szekeres (1961).
[17] Szekeres and Peters (2006).

trees with n vertices.[18] Rényi and Szekeres considered the much harder problem of counting the number of rooted trees with n vertices of height k.[19] They show that this number is given by

$$\sum_{\substack{m_1+\ldots+m_k=n-1\\ m_i\geq 0}} \frac{(n-1)!}{m_1!\ldots m_k!} m_1^{m_2}\ldots m_{k-1}^{m_k},$$

and use this expression to show that the average height of a tree with n vertices is approximately $(2\pi n)^{1/2}$.

Ahrens and Szekeres considered symmetric (v, k, λ) graphs.[20] These are graphs with v vertices in which every vertex is joined to exactly k others; further, given any two distinct vertices there are exactly λ vertices that are joined to both by edges. Such graphs are used to produce symmetric block designs, which are useful in statistics. Before their work, it was known that no $(v, k, 1)$ graphs exist, given a fixed λ greater than 1, there are at most a finite number of (v, k, λ) graphs; when λ is a power of a prime number, there is no more than one such graph. Ahrens and Szekeres showed that there is exactly one such graph and constructed it. Their argument used a cleverly chosen family of curves in the 3-dimensional affine space over the finite field with λ elements—the vertices of the graph are the curves in the affine space, and the edges of the graph join vertices for which the corresponding curves intersect.

The celebrated 'Four Colour Theorem' states that any map can be coloured using no more than four colours so that no countries with a common border have the same colour. This theorem is intimately connected with graph theory: a map may be turned into a graph by taking the countries to be the vertices, and linking those vertices that correspond to countries with a common border (not just a point). The 'chromatic number' of a graph is the smallest number of colours that are needed to paint all the vertices in such a way that no adjacent vertices have the same colour.

This theorem, which was proved by a computer examination of nearly two thousand different configurations in 1976 by Kenneth Appel and Wolfgang Haken,[21] had intrigued mathematicians for about one hundred and fifty years. It was known since the 1880s that if the result was false, then it would be possible to produce very symmetric graphs with several special properties (to be precise, simple, connected, bridgeless cubic graphs with chromatic index equal to 4) that could be drawn in the plane without any overlapping edges. Studying these graphs would help pinpoint the difficulties in proving the Four Colour Theorem, though their existence would not preclude the possibility of the theorem holding. The hunt for these graphs was on, and they became known as snarks (in honour of Lewis Carroll) because they were elusive. It is not surprising that one of the first nontrivial snarks was discovered by Szekeres.[22] Together with Herbert Wilf, Szekeres also discovered a useful inequality for the chromatic number of a graph.[23]

Other areas of discrete mathematics in which Szekeres worked include block designs and Hadamard matrices. He also drew pictures of Hadamard matrices that Judy Szekeres turned into embroidery; one of these still graces the art collection of the University of Wollongong.

Algebra and group theory

Several of George's outstanding early papers are in the area known as group theory. A group is a collection of elements that may be multiplied and inverted. For example, a block in the shape of a cube may be rotated around various axes in a total of 24 different ways, and the collection of these rotations is a group. Performing one rotation and then another gives the same result as performing a third rotation, known as the product of the first two; similarly, the inverse of a rotation is the rotation around the same axis but through the opposite angle. Groups of symmetries, such as this example, are of great importance in physics.

George's second paper is on group theory and applications in number theory;[24] it is notable as Erdős' first joint paper. Pure mathematicians (somewhat jokingly) are assigned Erdős numbers: Erdős himself has number 0; those who collaborated with him have Erdős number 1, those who collaborated with the collaborators of Erdős have Erdős number 2, and so on. So George had Erdős number 1, and he was the first number 1, of many hundreds.

George studied the problem of trying to classify all metabelian groups.[25] While this paper has not been widely cited, it illustrates George's interest in structure. At around the same time, he made considerable progress on the description of infinite torsion-free abelian groups,[26] extending the work of Ulm.[27] He continued to work in this area, but not everything was published. To quote George himself:

> If I am to choose one particular work that I regards as my best, I think that I would opt for a contribution … to the classification problem of representations of commutative rings which, for peculiar reasons, never got published. A few years later Professor I. M. Gelfand (with V. I. Ponomarev) in Moscow found the same results and I still get every year a New Year greeting card from Gelfand that I cherish more than if the article had been published.[28]

He returned to the subject of group theory several times over the course of his career, and worked on many different types of groups: finite and infinite, and abelian and nonabelian.

[18] Borchardt (1860). Cayley (1889).
[19] Rényi and Szekeres (1967).
[20] Ahrens and Szekeres (1969).
[21] Appel and Haken (1977).
[22] Szekeres (1973).
[23] Szekeres and Wilf (1967).
[24] Erdős and Szekeres (1934).
[25] Szekeres (1948*a*).
[26] Szekeres (1948*b*).
[27] Ulm (1933).
[28] *Curriculum vitae*, George Szekeres files, University of New South Wales Archives.

Partition theory

A partition of a whole number is a way of writing it as a sum of smaller numbers. For example, the number 5 may be partitioned in a total of seven different ways, namely, 5, $4+1$, $3+2$, $3+1+1$, $2+2+1$, $2+1+1+1$, and $1+1+1+1+1$. Partition theory is important in various branches of physics. Although this example makes partitions appear to be simple, it is practically impossible to write down the partitions of, say, 100, as there are several hundred million of these.

One of the great advances in the theory was the discovery by Hardy and Ramanujan of an asymptotic formula (that is, a formula that becomes relatively more accurate as n becomes larger) for $P(n)$, the number of partitions of the number n.[29] This was improved by Rademacher, who obtained an exact formula. Later, Erdős and Lehner[30] investigated $P(n,k)$, the number of partitions of n into at most k terms, and found an asymptotic formula for this when k is close to $n^{1/2}\log(n)$, the expected number of summands. Szekeres complemented this by obtaining an asymptotic formula for $P(n,k)$ (or more precisely a modified but equivalent version of this) when k is no greater than $2.7 \times n^{1/2}$;[31] in a second paper on the topic[32], he improved the result to find a formula for $P(n,k)$ when k is no greater that $C\,n^{1/2}$, for an arbitrary constant C. He also showed that $P(n,k)$ is increasing in k when k lies in a certain range. He further developed the topic with Klaus Roth, and, much later, by himself and then with many collaborators. It is noteworthy that in the first paper of this series, George thanks the CSIRO Mathematical Statistics section for help with a numerical computation.

We now give an application of partition theory. Szekeres and Guttmann considered triangular spiral self-avoiding random walks.[33] A random walk is the result of repeatedly choosing a direction 'at random' then taking a step in that direction. For instance, one might throw a die, and if the numbers 1 or 2 appear, turn right 90°, if the numbers 3 or 4 appear, do not change direction, and if the numbers 5 or 6 appear, turn left 90°. One then takes a step, and repeats the operation. Szekeres and Guttmann consider a random walk where one avoids all the places that one has already visited (this is the self-avoiding condition) and does not turn to the left (this makes the walk spiral); moreover, they allow turns of 60° and 120° (this condition characterises triangular walks). Self-avoiding random walks are used to model long chain molecules (polymers); the vertices represent the atoms of the molecule. Understanding the behaviour of self-avoiding random walks thus sheds light on the behaviour of certain materials. They show that the number of triangular self-avoiding n-step random walks is given by $C\exp(2\pi\sqrt{n})\log(n/12)/n^{13/4}$; the constant C depends on which angles one may turn through.

Number theory

In George's estimation, his most significant work in number theory is in the area of continued fractions.[34] By way of example, to find the continued fraction for the number π that appears whenever we consider circles we first note that π lies between 3 and 4, so that $\pi-3$ is between 0 and 1; next we consider $1/(\pi-3)$, which is $\sim$7.06264, and to observe that this lies between 7 and 8, so that $1/(\pi-3)-7$ also lies between 0 and 1; at step 3, we consider $1/(1/(\pi-3)-7)$, which is $\sim$15.96424, subtract 15 from this, and invert again. This process may be continued indefinitely, and if we do so, we deduce that we may write:

$$\pi = 3 + 1/(7 + 1/(15 + \ldots)).$$

This kind of expression is known as a continued fraction. If we forget the 'tail' of the fractions, then we find a sequence of rational numbers, namely 3, then $3 + 1/7$, then $3 + 1/(7 + 1/15) = 333/106$, and so on. These rational numbers, 3, 22/7, 333/106, 355/113, and so on, are good approximations to π: the first was used in the Bible to describe Solomon's Temple, the second is commonly used in schools as it is easy to use and is accurate to two decimal places, while the fourth was known to the Chinese some fifteen hundred years ago, and is accurate to six decimal places.

Szekeres' work on multidimensional continued fractions aims to find simultaneous rational approximations for several real numbers.[35] These may then be used to looked for rational relationships between irrational numbers. Several of his PhD students continued this study into the 1970s and 1980s.

Mathematical analysis

It is said that the area of mathematical analysis is characterised by its willingness to treat infinite sequences and infinite sums, known as series. George Szekeres was an expert in the art of using infinite series to extract information. One very early paper in analysis explores the coefficients of a power series.[36] His remarkable proficiency with infinite series underpinned many of his other papers, both on analysis and on other topics, such as partition theory.

One of the ongoing themes of George's research in analysis is the iteration of functions. For instance, if $f(x) = 2x + 1$ and $g(y) = 3y + 4$, then the iterated or composed function 'g of f' or $g \circ f$ is given by

$$g(f(x)) = 3(2x + 1) + 4 = 6x + 7.$$

More generally, we may define a sequence of functions f_σ, where $\sigma = 1, 2, 3, \ldots$, by setting $f_1 = f$, $f_2 = f \circ f_1$, $f_3 = f \circ f_2$,

[29] Hardy and Ramanujan (1918).
[30] Erdős and Lehner (1941).
[31] Szekeres (1951).
[32] Szekeres (1953).
[33] Szekeres and Guttmann (1987).
[34] *Curriculum vitae*, George Szekeres files, University of New South Wales Archives.
[35] Szekeres (1971).
[36] Szekeres (1950).

and so on. In even greater generality, we define a family of iterates of a function f to be a collection of functions f_σ, indexed by a real or complex parameter σ, such that $f_{\sigma+\tau} = f_\sigma \circ f_\tau$. Iterated functions and functional equations of Schroeder and Abel type are important in statistics, where they are used to describe a branching stochastic process. One of the fundamental questions is whether all functions may be realised as compositions of other functions, and if not, how do we recognise when a function is a composed function. For a long time, iteration of functions had been considered in the context of complex variables, but in 1949, Kneser showed that the exponential function, one of the most basic functions of mathematics, is iterated when considered as a function of a real variable;[37] this had been long known to be impossible when the exponential is considered as a function of a complex variable.

In an imposing work (certainly one of his best), Szekeres developed the theory of iteration for functions of a real variable, and functions defined on sectors of circles in the complex plane.[38] The Schroeder functional equation looks for eigenfunctions of the operator of composition with a given function f, that is, it looks for eigenfunctions χ such that $\chi \circ f = \alpha\chi$, where α is the eigenvalue. When such a function χ may be found, then iterates of f may often be determined using the formula $f_\sigma = \chi^{-1} \circ (\alpha^\sigma \chi)$, where the expression in parentheses is the function χ multiplied by the scalar α^σ; further, χ may often be found as a limit $\lim_{n\to\infty} \alpha^n f_n$. In this paper, George makes these 'often' statements precise, and puts the connection between the iterates and the functional equation on solid ground. Other functional equations appear along the way, in particular Abel's equation $\lambda \circ f = \lambda - 1$.

George returned to the study of functional equations and iteration of functions many times during his career, apparently often stimulated by results of Noel Baker,[39] who in turn worked on extending George's discoveries. Several of George's papers on iteration, including the first, mention work of Baker explicitly, and one of Szekeres' very last papers was a contribution to a volume in Baker's honour.[40]

Numerical integration

One part of classical numerical integration is about finding ways to evaluate (to whatever degree of precision is required) the area under the graph of a given function. This goes back to Newton, who observed that the area may be divided into many small 'near-rectangles', with possibly curved arcs on top, and that one could find better estimates for their areas if one considered the curvature of the arc on top in the most appropriate way. George, especially with his student Tom Sag, was interested in trying to find volumes under graphs of functions of several variables, over regions of peculiar shapes. This rapidly becomes much more difficult than Newton's original problem. The strategy that George and Tom adopted was to find a suitable transformation in the base of the region so that this becomes circular, where additional symmetries can be employed to find very good numerical approximations.[41] It seems that the method of Sag and Szekeres was not widely adopted; it was not cited much before 2000 but has been cited considerably since, so perhaps this will change.

Mathematical physics

Most of George's work in (mathematical) physics falls into three main categories: his work on singularities, his gravitation theory that involved a cosmic time, and his investigation of a spinor-connection approach to theoretical physics.

George's best-known paper in relativity appeared in a volume dedicated to the geometer Ottó Varga.[42] It was perhaps best known by rumour until its republication in 2002,[43] and turns out to be known for what George himself thought were the wrong reasons. His purpose, as he himself said, was to 'expound [his] own definition of a singularity', and the application to the Schwarzschild metric he saw as a mere illustration. That it led to the maximal analytic extension of the Schwarzschild solution, now known as the Kruskal–Szekeres metric, was a happy accident. At the time, the notion of a singularity was very poorly understood, and the wording of the paper clearly indicates the then-prevalent uncertainty about what constituted a singularity. Indeed, relativists had yet to fully appreciate the difference between the coordinate singularity now called the event horizon (which the Kruskal–Szekeres transformation regularises) and more serious curvature singularities. Nevertheless, although Martin Kruskal's paper on the transformation had the benefit of a wider readership, the ideas involved were part of the ignition of a serious consideration of singularities in relativity throughout the 1960s, which led to, for example, the famous singularity theorems of Hawking and Penrose. Work on singularities continues to the present day, and are all based on the idea that singularities are given by the behaviour of geodesics (the paths of free particles). George's paper may well be the first time this idea was seriously considered.

George's first papers dealing with general relativity were motivated by a desire to create a theory that included a cosmologically preferred state of motion, such as might be given by the universe's underlying bulk matter. In essence, he accepted what is often now called the 'semi-strong' principle of equivalence (spacetime is locally the spacetime of special relativity), but rejected the 'very strong' principle (all reference frames are equivalent).

The first paper of this group introduces this theory,[44] postulating an absolute cosmic time. George derived the field equations of this theory and worked out some consequences, such as the version in his theory of the standard cosmological model and the Schwarzschild black-hole. In a historically curious consequence, given the result described two paragraphs ago, the latter solution does not have a singularity. George also worked out detailed consequences

[37] Kneser (1949).
[38] Szekeres (1958).
[39] Baker (1958).
[40] Szekeres (2008).
[41] Sag and Szekeres (1964).
[42] Szekeres (1960).
[43] Szekeres (2002).
[44] Szekeres (1955).

of his models, such as the energy density of the Universe, and the perihelion precession of satellites. The latter leads to an effect somewhat smaller than that predicted by general relativity. At the time of the paper, the known measurement of the precession of Mercury were consistent with George's prediction (with suitable assumptions), but his theory is not consistent with more recent measurements.

In the second paper, George looked at the effect his cosmologically mandated absolute motion would have on a test particle.[45] In a series of calculations that even today look heroic, he came to the conclusion that with the data available in 1955, the absolute velocity of the solar system must be less than 100 km per second: about half the sun's orbital velocity in our galaxy.

In the third paper, George and his collaborator Wallace Kantor investigated other possible theories, consistent with what they called a 'gauge postulate'.[46] Much as in the first paper of this group, they derived field equations and investigated the cosmological and spherically symmetric solutions. In this case, the latter solutions turned out to be identical with those of General Relativity.

George's next foray in the field of mathematical physics was quite different in character.[47] In it he was concerned with the basic mathematical structure of spacetime and attempted to base the geometry on a spinor bundle. This work was later extended by George and his students.[48] Connections on spinor bundles are an active area of study, but this work seems to remain largely unnoticed.

Numerical experimentation

As already mentioned, George was a pioneer in the use of numerical work to formulate and test conjectures. His work was often informed by experiment. K. T. Briggs wrote to J. J. O'Connor and E. F. Robertson, the authors of the online biography:

> I would thus describe George as a pioneer of experimental mathematics—he saw the potential of the computer, particularly in testing conjectures, very early.[49]

Examples of this include his work on continued fractions and on partitions.

George Szekeres the mentor

George and Esther (and other members of their Budapest circle) were always very interested in problems that could be put to high school students or university undergraduates. Several of George's publications either pose or solve such problems. For example, Paul Erdős asked about how many numbers less that a given number *n* are either divisors or multiples of certain other numbers, $a_1, a_2, ..., a_k$, and George showed that the problem may be reduced to the special case where the numbers $a_1, a_2, ..., a_k$ are the first *k* prime numbers, that is, 2, 3, 5, 7, and so on.[50] Problems of this nature are now widely used to provide mathematical stimulation for very talented high school students all around the world.

In addition, George spent a lot of time with bright young high school students in connection with the mathematics competitions with which he was involved, and engaged with bright young undergraduates, including David Harvey,[51] now an associate professor at UNSW Sydney.

The following list of his post-graduate students is arguably not complete, as he often advised students unofficially: Bill Atterton (PhD), Noel Baker (MSc), Michael Cullinan (PhD), Phil Diamond (PhD), Mary Ruth Freislich (MA), John Giles (PhD), Jack Gray (PhD), Sam Krass (PhD), John Lynch (PhD), John Mack (PhD), Roman Matlak (PhD), James Michael (PhD), Laurence Misaki (MSc), Bob Perry (PhD), Lindsay Peters (PhD), Tom Sag (MSc), Cedric Schubert (MSc), John Schutz (PhD), Geoffrey Smith (PhD), Peter Trotter (PhD), Alf van der Poorten (PhD), Peter Wark (MSc). In addition, he supervised many undergraduate projects, and often provided support and suggestions to his colleagues at the Universities of Adelaide and New South Wales and elsewhere. The first-named author of this article has certainly profited from his advice.

Acknowledgements

It is a pleasure to thank Judy and Peter Szekeres for discussions, as well as many of our colleagues who have helped in different ways. The article of Giles and Seberry was useful.[52] David Tacon offered many recollections of his conversations with George, and Andrew Jakubowicz pointed the authors at some relevant material about Shanghai. Eugene Seneta was helpful in many ways, and Gábor Székely and Vera Turán-Sós provided information about the Szekeres' Hungarian connections. The referees offered many useful suggestions. While writing this, we consulted documents held by the Australian National Archives, by the University of New South Wales, by the Sydney Jewish Museum, and the University of Adelaide.

Conflicts of interest

The authors declare no conflicts of interest.

References

Ahrens, R. W., and Szekeres, G. (1969) On a combinatorial generalization of 27 lines associated with a cubic surface, *Journal of the Australian Mathematical Society*, **10**, 485–492. doi:10.1017/S144678870000776X

Appel, K., and Haken, W. (1977) Solution of the Four Color Map Problem, *Scientific American*, **237**, 108–121. doi:10.1038/SCIENTIFICAMERICAN1077-108

Baker, I. N. (1958) Zusammensetzungen ganzer Funktionen, *Mathematische Zeitschrift*, **69**, 121–163. doi:10.1007/BF01187396

Borchardt, C. W. (1860) Über eine Interpolationsformel für eine Art Symmetrischer Functionen und über Deren Anwendung, *Mathematische*

[45] Szekeres (1956).
[46] Kantor and Szekeres (1956).
[47] Szekeres (1957).
[48] Lynch (1985). Szekeres (1975). Szekeres and Peters (2008).
[49] O'Connor and Robertson (2006).
[50] Erdős and Szekeres (1953).
[51] Harvey (2002).
[52] Giles and Seberry Wallis (1976).

Abhandlungen der königlicher Akademie der Wissenschaften zu Berlin, 1–20.

Butt, P. (2012) *Who Killed Dr Bogle & Mrs Chandler?* London–Chatswood, NSW.

Cayley, A. (1889) A theorem on trees, *Quarterly Journal of Pure and Applied Mathematics*, **23**, 376–378.

Cohen, G. L. (2006) *Counting Australia In. The People, Organisations and Institutions of Australian Mathematics*, Sydney.

Cowling, M. G. (2005*a*) Obituary George and Esther Szekeres, *Gazette of the Australian Mathematical Society*, **32**, 221–225.

Cowling, M. (2005*b*) *Sydney Morning Herald* (Sydney), 7 November, 28.

Erdős, P., and Lehner, J. (1941) The distribution of the number of summands in the partitions of a positive integer, *Duke Mathematical Journal*, **8**, 335–345.

Erdős, P., and Szekeres, G. (1934) Über die Anzahl der Abelschen Gruppen gegebener Ordnung und über ein verwandtes zahlentheoretisches Problem, *Acta Litterarum ac Scientiarum: Regiae Universitatis Hungaricae Francisco-Josephinae. Sectio Scientiarum Mathematicarum*, **7**, 95–102.

Erdős, P., and Szekeres, G. (1935) A combinatorial problem in geometry, *Compositio Mathematica*, **2**, 463–470.

Erdős, P., and Szekeres, G. (1953) Problems and solutions: advanced problems: solutions 4352, *The American Mathematical Monthly*, **60**, 717–718. doi:10.2307/2307167

Erdős, P., and Szekeres, G. (1961) On some extremum problems in elementary geometry, *Annales Universitatis Scientiarum Budapestinensis de Rolando Eötvös Nominatae: Sectio Mathematica*, **3–4**, 53–62.

Giles, J. R., and Seberry Wallis, J. (1976) George Szekeres: with affection and respect, *Journal of the Australian Mathematical Society. Series A*, **21**, 385–392. doi:10.1017/S1446788700019212

Hanák, P. (1999) *The Garden and the Workshop*, Princeton.

Hardy, G. H., and Ramanujan, S. (1918) Asymptotic formulæ in combinatory analysis, *Proceedings of the London Mathematical Society: Second Series*, **17**, 75–115. doi:10.1112/PLMS/S2-17.1.75

Hargittai, I. (2006) *The Martians of Science: Five Physicists Who Changed the Twentieth Century,* New York.

Harvey, D. M. (2002) 'George Szekeres Counts to Ninety', in 'AustMS Library', https://www.austms.org.au/Jobs/Library2.html, viewed December 2017.

Hoffman, P. (1998) *The Man Who Loved Only Numbers: the Story of Paul Erdős and the Search for Mathematical Truth*, New York.

Kantor, W., and Szekeres, G. (1956) Cosmic time and the field equations of general relativity, *Physical Review, II Series*, **104**, 831–834.

Kneser, H. (1949) Reelle analytische Lösungen der Gleichung $\varphi(\varphi(x)) = e^x$ und verwandter Funktionalgleichungen, *Journal für die reine und angewandte Mathematik: Crelle's Journal*, **187**, 56–67.

Kranzler, D. (1976) *Japanese, Nazis, and Jews: the Jewish Refugee Community of Shanghai, 1938–1945*, New York.

Lynch, J. T. (1985) Four forces and spinor connection in general relativity, *Physical Review. D*, **31**, 1287–1295. doi:10.1103/PHYSREVD.31.1287

Marton, K. (2006) *The Great Escape: Nine Jews Who Fled Hitler and Changed the World,* New York.

'MATP' (2005) *Daily Telegraph* (Sydney), 31 August, 38.

Morgan, J. (2005*a*) *Advertiser* (Adelaide), 31 August, 3.

Morgan, J. (2005*b*) *Herald-Sun* (Melbourne), 31 August, 4.

O'Connor, J. J., and Robertson, E. F. (2006) 'George Szekeres', in 'MacTutor History of Mathematics archive', http://www-history.mcs.st-andrews.ac.uk/Biographies/Szekeres.html, viewed October 2017.

Pan, Guang (ed) (1995) *The Jews in Shanghai*, Shanghai.

Ramsey, F. P. (1929) On a problem in formal logic, *Proceedings of the London Mathematical Society*, **30**, 264–286.

Rényi, A., and Szekeres, G. (1967) On the height of trees, *Journal of the Australian Mathematical Society*, **7**, 497–507. doi:10.1017/S1446788700004432

Sag, T. W., and Szekeres, G. (1964) Numerical evaluation of high-dimensional integrals, *Mathematics of Computation*, **18**, 245–253. doi:10.1090/S0025-5718-1964-0165689-X

Schechter, B. (2000) *My Brain is Open: the Mathematical Journeys of Paul Erdős,* New York.

Szekeres, G. (1948*a*) On a certain class of metabelian groups, *Annals of Mathematics. Second Series*, **49**, 43–52. doi:10.2307/1969112

Szekeres, G. (1948*b*) Countable abelian groups without torsion, *Duke Mathematical Journal*, **15**, 293–306. doi:10.1215/S0012-7094-48-01529-4

Szekeres, G. (1950) The asymptotic behaviour of the coefficients of certain power series, *Acta Scientiarum Mathematicarum*, **12**, 187–198.

Szekeres, G. (1951) An asymptotic formula in the theory of partitions, *The Quarterly Journal of Mathematics, Oxford: Second Series*, **2**, 85–108. doi:10.1093/QMATH/2.1.85

Szekeres, G. (1953) Some asymptotic formulae in the theory of partitions: II, *The Quarterly Journal of Mathematics, Oxford: Second Series*, **4**, 96–111. doi:10.1093/QMATH/4.1.96

Szekeres, G. (1955) New formulation of the general theory of relativity, *Physical Review, II: Series*, **97**, 212–223.

Szekeres, G. (1956) Ether drift and gravitational motion, *Physical Review. II: Series*, **104**, 1791–1798.

Szekeres, G. (1957) Spinor geometry and general field theory, *Journal of Mathematics and Mechanics*, **6**, 471–517.

Szekeres, G. (1958) Regular iteration of real and complex functions, *Acta Mathematica*, **100**, 203–258. doi:10.1007/BF02559539

Szekeres, G. (1960) On the singularities of a Riemannian manifold, *Publicationes Mathematicae*, **7**, 285–301.

Szekeres, G. (1971) Multidimensional continued fractions, *Annales Universitatis Scientiarum Budapestinensis de Rolando Eötvös Nominatae: Sectio Mathematica*, **13**, 113–140.

Szekeres, G. (1973) Polyhedral decompositions of cubic graphs, *Bulletin of the Australian Mathematical Society*, **8**, 367–387. doi:10.1017/S0004972700042660

Szekeres, G. (1975) Cosmology with spinor connection, *Computers & Mathematics with Applications*, **1**, 345–350. doi:10.1016/0898-1221(75)90034-6

Szekeres, G. (2002) On the singularities of a Riemannian manifold, *General Relativity and Gravitation*, **34**, 2001–2016. doi:10.1023/A:1020744914721

Szekeres, G. (2008) 'Abel's functional equation and its role in the problem of croissance régulière' in *Transcendental Dynamics and Complex Analysis*: *a Tribute to Noel Baker*, eds. P. J. Rippon and G. M. Stallard, Cambridge, pp. 447–451.

Szekeres, G., and Guttmann, A. J. (1987) Spiral self-avoiding walks on the triangular lattice, *Journal of Physics. A, Mathematical and General*, **20**, 481–493. doi:10.1088/0305-4470/20/2/033

Szekeres, G., and Peters, L. (2006) Computer solution to the 17-point Erdős–Szekeres problem, *The ANZIAM Journal*, **48**, 151–164. doi:10.1017/S144618110000300X

Szekeres, G., and Peters, L. (2008) Space-time structure and spinor geometry, *The ANZIAM Journal*, **50**, 143–176. doi:10.1017/S1446181109000108

Szekeres, G., and Wilf, H. S. (1967) An inequality for the chromatic number of a graph, *Journal of Combinatorial Theory*, **4**, 1–3. doi:10.1016/S0021-9800(68)80081-X

T.-Sós, V., and Laczkovich, M. (2004/2005) Emlékezés Klein Eszterre és Szekeres Györgyre, *Mathematikai Lapok*, **12**(2), 9–11.

Ulm, H. (1933) Zur Theorie der abzählbar-unendlichen Abelschen Gruppen, *Mathematische Annalen*, **107**, 774–803. doi:10.1007/BF01448919